

Price Level and Inflation Dynamics in Heterogeneous Agent Economies

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Abstract

We study price level dynamics in a heterogeneous-agent incomplete-market economy with nominal government debt and flexible prices. Unlike in representative agent economies, steady-state equilibria exist when the government runs persistent deficits, provided the level of deficits is not too large. We quantify the maximum sustainable deficit for the US and show that it is lower under more redistributive tax and transfer systems. With constant primary deficits, there exist two steady-states, and the price level and inflation are not uniquely determined. We describe alternative policy settings that deliver uniqueness. We conduct quantitative experiments to illustrate how redistribution and precautionary saving amplify price level increases in response to fiscal helicopter drops, deficit expansions, and loose monetary policy. We show that rising primary deficits can account for a decline in the long-run real interest rate, leading to permanently higher inflation. Our work highlights the role of household heterogeneity and market incompleteness in determining inflation dynamics.

Keywords: Fiscal theory of the price level, Heterogeneity, Incomplete markets, Inflation, Precautionary saving, Redistribution, Sustainable deficit.

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1 Introduction

We develop a framework to study the causes and consequences of price level dynamics in an economy with three features: (i) a fiscal authority issues nominal debt to finance committed real expenditures and transfers to households; (ii) a monetary authority sets the short-term nominal rate on government debt; (iii) financial markets are incomplete, so households have a precautionary motive to accumulate savings in order to self-insure against idiosyncratic income risk.

Our interest in economies with the first two features is motivated by institutional arrangements in the real world. Such economies have been extensively studied, most recently under the label *Fiscal Theory of the Price Level* (FTPL).¹ They have also been a useful lens to analyze the most recent bout of inflation that followed large expansions in government borrowing, a global supply shock due to the COVID-19 pandemic, and sharp interest rate movements by central banks around the world. This literature has focused almost entirely on representative agent economies.

Our motivation for extending this analysis to “Bewley” economies (Bewley, 1987) with heterogeneous agents and incomplete markets is three-fold. First, heterogeneous agent models generate consumption responses to income and interest rates that are consistent with the vast body of micro-economic evidence on the joint dynamics of household income and spending.² This property is important because household spending pressure is a key force shaping inflation and interest rates in equilibrium.

Second, household heterogeneity has played an important role in both the drivers and consequences of the current inflationary episode. Governments issued vast quantities of new debt to finance transfers that were targeted to certain groups of households. The ongoing spending pressures that are leading many government to run persistent deficits are also highly targeted. Quantitative heterogeneous agent models are a natural environment to study the implications of such interventions, as well as the distributional effects of shocks and subsequent policy responses.

¹The FTPL literature, which has its roots in Sargent and Wallace (1981) and builds on Leeper (1991), Sims (1994), Woodford (1995) and Cochrane (1998) is too vast to cite in full. See the handbook chapter by Leeper and Leith (2016) and book by Cochrane (2023) for a synthesis of the reach of FTPL models.

²See for example the review article by Kaplan and Violante (2022).

Third, working in a heterogeneous-agent incomplete-market setting also overcomes a limitation of representative agent FTPL models that makes their application to current macroeconomic conditions problematic. Standard representative agent models require governments to run positive primary surpluses in expectation at all points in time. However, in recent decades the US has consistently run primary deficits, and the fiscal positions of the US and many other developed economies look unlikely to return to surpluses anytime soon.³ Heterogeneous agent versions of these models offer a natural setting in which to study price level dynamics with persistent primary deficits. In these versions, the real return on government debt r is less than the growth rate of the economy g , which is also a feature of recent macroeconomic conditions.

This motivation leads us to start building a bridge between the well-studied representative-agent FTPL and workhorse heterogeneous-agent models in the tradition of [Bewley \(1987\)](#), [Imrohoroglu \(1989\)](#), [Huggett \(1993\)](#) and [Aiyagari \(1994\)](#). In this paper, we take a first step by focusing on flexible-price economies.⁴

Theoretical Analysis. We begin by analyzing an endowment economy in which the government runs positive primary surpluses and $r > g$. Here, the conditions on monetary and fiscal policy for the price level and inflation to be uniquely determined are essentially unchanged from corresponding representative agent economies. There are, however, important quantitative differences that reflect the role of precautionary savings. Unlike in the representative agent economy, in the heterogeneous agent economy changes in fiscal policy lead to movements in the real interest rate. This is because a change in either the level of debt, or the size and distribution of surpluses alters the overall demand for savings among households. For a given setting of monetary policy, these different real rate dynamics imply different paths of inflation. It also means that there are non-trivial inflation dynamics following a one-time fiscal helicopter drop, and that the path of inflation depends on the targeting of the fiscal injection. We use a modified representative agent model with bonds in the

³With the exception of 1998-2001, the US has not run a primary surplus since 1970. See Series FYFSD from FRED, Federal Reserve Bank of St. Louis, <https://fred.stlouisfed.org>. Moreover, the May 2023 10-year budget projections of the Congressional Budget Office (CBO) estimate that deficits will remain negative at least until 2033: <https://www.cbo.gov/data/budget-economic-data>

⁴In ongoing work we extend to economies with nominal rigidities. See [Kaplan et al. \(2023\)](#).

utility function to provide intuition for these forces. We then analyze the same heterogeneous-agent economy but with a government that runs a constant primary deficit and $r < g$. We show that, as long as the level of deficits is not too large, equilibria with a finite price level where debt is valued exist. The maximum possible level of deficits is decreasing in the amount of redistribution implicit in the tax and transfer system: more redistribution reduces aggregate precautionary saving and increases real interest payments on debt. For lower levels of deficits, there are generically two steady-states. Thus, without additional assumptions, standard FTPL arguments do not uniquely pin down the price level or the path of inflation. The steady-states are Pareto ranked, with the high debt, high interest rate, low inflation steady-state delivering larger welfare to every household. The low inflation steady-state is saddle-path stable: there is a unique initial price level and subsequent path of inflation and real rates leading to that steady-state. The high inflation steady-state is locally stable: there is a continuum of initial price levels that support paths of inflation leading to that steady-state.

We discuss various extensions that deliver a unique prediction for the price level and inflation. First, we propose modifications to the model that eliminate the high inflation steady-state altogether, leaving only a unique saddle-path stable steady-state. These modifications include (i) fiscal reaction rules that allow the level of surpluses to respond to deviations of real debt or the real rate from steady-state; and (ii) the introduction of a foreign sector with a relatively inelastic demand for domestic government debt. Second, we propose a policy environment in which the central bank successfully coordinates private sector expectations about long-run inflation. By anchoring long-run inflation expectations to be consistent with the saddle-path stable steady-state, uniqueness is also achieved in the short run, because all the equilibria that converge to the high inflation steady-state are eliminated.

With uniqueness of equilibria in hand, we move to the quantitative analysis.

Quantitative Policy Messages. In the quantitative part of the paper, we conduct a series of experiments to illustrate lessons for policy that emerge in the heterogeneous agent setting, but are concealed in more traditional representative agent FTPL environments.

First, we consider the effects of permanently increasing deficits. We calculate

that if the government were to permanently increase lump sum transfers to households without raising taxes, the largest sustainable primary deficit would be 4.6% of GDP, or 40% higher than current levels. The maximum sustainable deficit depends on the degree of social insurance: expanding deficits in a more progressive manner implies lower maximum deficits. The reason is that tax systems that provide more social insurance weaken precautionary savings, thus lowering household demand for government debt. More progressive tax systems therefore reduce fiscal space.

A permanently higher deficit is associated with a lower steady-state real interest rate and less real government debt, as well as a higher long-run inflation rate for a given nominal rate target. This is because a larger deficit must be funded by larger real interest receipts, which require a more negative real rate. The heterogeneous agent framework thus offers an alternative interpretation of discussions around secular stagnation by highlighting the connection between a rising primary deficit, falling real rates and rising inflation.

Next, we study the effects of issuing new debt while holding primary deficits constant: a fiscal helicopter drop. We consider a helicopter drop of around 16% of annual GDP, roughly the size of the fiscal expansion in the US over the course of the COVID-19 pandemic. Consistent with the representative agent experiments in [Cochrane \(2022\)](#), we find that this generates an immediate jump in the price level. However, relative to the representative agent benchmark, in our economy there is an additional 30% initial increase in the price level. This amplification is driven by redistribution and heterogeneity of marginal propensities to consume (MPC): in the heterogeneous agent economy, the dilution of nominal debt entails large amounts of redistribution from wealthy to poor households. This reallocation of wealth generates upward pressure on consumption, which increases real rates and interest payments on government debt, thereby causing a larger initial jump in the price level. A targeted helicopter drop such as that implemented in the US, which targets high MPC households, fuels additional short-term inflationary pressures.

Lastly, we study the effects of purely redistributive policies that hold both debt and deficits constant, and show that budget neutral redistribution is inflationary. We illustrate these effects by way of numerical experiments in which the government levies a one-time wealth tax on household in the top percentiles of the wealth distribution,

and redistributes the proceeds lump-sum to households in the bottom half of the wealth distribution. As with the fiscal helicopter drop, real redistribution towards high MPC households leads to a temporarily higher real interest rate and a downward revaluation of real assets through a jump in the price level.

Related Literature. Our paper belongs to a small but growing literature that moves beyond the representative agent model and explores the FTPL with incomplete markets. [Bassetto and Cui \(2018\)](#) show that a model of overlapping generations and a model in which government debt provides special liquidity services can give rise to multiple steady-states in which the real interest rate on government debt is below the growth rate of output. They emphasize that the FTPL can fail to yield price level determinacy in these settings. [Brunnermeier et al. \(2020, 2022\)](#), [Miao and Su \(2021\)](#) and [Amol and Luttmer \(2022\)](#) all study models with idiosyncratic risk in the rate of return on capital, and explore settings for fiscal policy that can establish price level uniqueness in low interest rate environments.

Our work differs from these papers in three respects. First, we investigate the implications of the FTPL in a [Bewley \(1987\)](#) economy in which market incompleteness arises from uninsurable labor income risk.⁵ In doing so, we emphasize the importance of MPC heterogeneity in driving price level and inflation dynamics. Second, we explore a wide class of fiscal, monetary, and institutional specifications and show how they lead to price level uniqueness in models where the government runs persistent primary deficits. Third, we quantitatively explore the response of economic aggregates to unanticipated shocks in low-interest rate economies with persistent deficits. To the best of our knowledge, the messages we deliver about the role of precautionary savings and MPC heterogeneity in driving price level, inflation and real rate dynamics in this class of economies are novel.⁶

Our work also relates to the literature that studies the implications of low interest

⁵[Hagedorn \(2021\)](#) also explores price-level determination in a “Bewley” economy with nominal government debt, but focuses on a different class of fiscal policies outside FTPL.

⁶Some qualitative aspects of our analysis, such as equilibrium multiplicity with deficits, share features with certain monetarist economies. See, for example, Chapter 18 of [Ljungqvist and Sargent \(2018\)](#).

rate environments for government borrowing (Aguiar et al., 2021; Blanchard, 1985, 2019; Cochrane, 2021; Kocherlakota, 2023; Mehrotra and Sergeyev, 2021; Reis, 2021). This body of work emphasizes that the government can roll over debt indefinitely when the real interest rate on government debt is below the growth rate of the economy.⁷ We show that this stark conclusion is correct only up to a limit: there is a finite upper bound on primary deficits for there to exist an equilibrium in which government debt is valued. We quantify this bound in our calibrated model for the U.S. economy and illustrate how it depends on the level of uninsurable income risk and on the degree of fiscal redistribution.⁸

Finally, our work highlights the importance of household heterogeneity in determining interest rates and inflation. As such, it relates to work that explores the distributional consequences of monetary policy and inflation (Doepke and Schneider, 2006; Coibion et al., 2017; McKay and Wolf, 2023) and the role of agent heterogeneity in amplifying economic outcomes (Auclert et al., 2018; Kaplan et al., 2018; Auclert, 2019). In particular, we show that unanticipated changes in the price level can give rise to non-trivial, persistent dynamics in the real interest rate and inflation due to heterogeneous wealth effects across the distribution.

2 Model Environment

2.1 Households

Demographics. Time is continuous and is indexed by $t \geq 0$. The economy is populated by a continuum of households indexed by $j \in [0, 1]$.

Endowments. Real aggregate output y_t is exogenous and grows at a constant rate $g \geq 0$. Household j receives a stochastic share z_{jt} of aggregate output. The shares z_{jt} are independent across households and a law of large numbers holds so that there is no economy-wide uncertainty,

$$\int_{j \in [0,1]} z_{jt} dj = 1 \text{ for all } t \geq 0. \quad (1)$$

⁷Angeletos et al. (2023) show that in non-Ricardian economies with nominal rigidities, it is possible for government deficits to be self-financing, even when $r > g$.

⁸The insight that the size of fiscal space depends on the use the government makes of this space is shared by Mian et al. (2021a) and Amol and Luttmer (2022). However, precautionary saving plays no role in the two-agent model of Mian et al. (2021a), and redistribution plays no role in the model of Amol and Luttmer (2022) where all agents have the same MPC. In our economy, the strength of consumption insurance and fiscal redistribution forces determined endogenously in equilibrium.

In our baseline model we assume that z_{jt} follows an N -state Poisson process with switching intensities $\lambda_{z,z'}$. The lowest value of the endowment share \underline{z} is strictly positive, $\underline{z} > 0$, from which it follows that the natural borrowing limit is below zero.⁹ Not For Publication Appendix G presents a model in which z_{jt} follows a diffusion.

Assets. Households trade a short-term risk-free bond that yields a nominal flow return i_t . We denote the nominal bond holdings of household j at time t by A_{jt} . This asset is the unit of account in the economy, and we let P_t denote the price of output in terms of this short-term bond.

Preferences. Households take the path of aggregate variables $\{P_t, i_t, y_t\}_{t \geq 0}$ as given and choose real consumption flows \tilde{c}_{jt} to maximize

$$\mathbb{E}_0 \int e^{-\tilde{\rho}t} \frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma} dt \quad (2)$$

with $\gamma \geq 0$, where the expectation is taken over the idiosyncratic endowment process z_{jt} . We denote the household's discount rate by $\tilde{\rho} > 0$.

Nominal Household Budget Constraint. Initial nominal assets A_{j0} are given. For $t > 0$, households face a flow budget constraint

$$dA_{jt} = [i_t A_{jt} + (z_{jt} - \tau_t(z_{jt})) P_t y_t - P_t \tilde{c}_{jt}] dt. \quad (3)$$

The path of tax and transfer functions $\tau_t(z)$ is set by the fiscal authority and is described in more detail below. Nominal savings dA_{jt} are equal to the sum of asset income $i_t A_{jt}$ and endowment income net of taxes and transfers $(z_{jt} - \tau_t(z_{jt})) P_t y_t$, minus consumption expenditures $P_t \tilde{c}_{jt}$. In our baseline model we assume that households cannot borrow $A_{jt} \geq 0$, but we relax this assumption in Section 5. Online Appendix E.1 contains an analysis of the model with borrowing.

Price Level and Inflation. Since this is a flexible-price economy, the price level P_t may exhibit jumps. For ease of notation and exposition, we restrict the price level to jump only at $t = 0$, after which it follows a deterministic path.¹⁰ Since there is no

⁹In our quantitative experiments in which we allow for borrowing, the interest rate on loans is always positive so the natural debt limit is well-defined.

¹⁰Studying perfect foresight solutions with a single probability-zero jump at time zero is commonly maintained in FTPL models (Leeper, 1991; Sims, 2011; Cochrane, 2018). The absence of aggregate uncertainty implies that the price level cannot exhibit jumps for $t > 0$ in discrete time, representative

intrinsic (i.e., fundamental) aggregate uncertainty, this implies perfect foresight over aggregate variables for $t > 0$. For $t > 0$, we define the inflation rate by

$$\frac{dP_t}{P_t} = \pi_t dt. \quad (4)$$

De-trended Real Household Budget Constraint. We denote de-trended real assets and de-trended real consumption as

$$a_{jt} := \frac{A_{jt}}{P_t y_0 e^{gt}} \quad c_{jt} := \frac{\tilde{c}_{jt}}{y_0 e^{gt}} \quad (5)$$

For $t > 0$, we can re-write the nominal budget constraint (3) in de-trended real terms:

$$da_{jt} = [r_t a_{jt} + z_{jt} - \tau_t(z_{jt}) - c_{jt}] dt \quad (6)$$

where

$$r_t := i_t - \pi_t - g \quad (7)$$

is the growth-adjusted real rate. At $t = 0$, de-trended real assets a_{j0} are given by the ratio of initial nominal assets A_{j0} to the endogenous initial price level P_0 .

Relative Asset Holdings. Let A_t and a_t denote aggregate nominal and aggregate de-trended real household assets, respectively:

$$A_t := \int_{j \in [0,1]} A_{jt} dj \quad a_t := \int_{j \in [0,1]} a_{jt} dj$$

We denote the share of assets held by household j at time t by $\omega_{jt} := \frac{A_{jt}}{A_t} = \frac{a_{jt}}{a_t}$, with

$$\int_{j \in [0,1]} \omega_{jt} dj = 1 \text{ for all } t \geq 0. \quad (8)$$

Recursive Formulation of Household Problem. Given paths of real rates r_t and taxes τ_t , the household problem can be expressed recursively via the Hamilton-Jacobi-Bellman Equation (HJB)

$$\begin{aligned} \rho V_t(a, z) - \partial_t V_t(a, z) &= \max_c \frac{c^{1-\gamma}}{1-\gamma} + \partial_a V_t(a, z) [r_t a + z - \tau_t(z) - c] \\ &\quad + \sum_{z' \neq z} \lambda_{z, z'} [V_t(a, z') - V_t(a, z)], \end{aligned} \quad (9)$$

agent FTPL models (Cochrane, 2023).

together with the boundary condition $\partial_a V_t(0, z) \geq (z - \tau_t(z))^{-\gamma}$ that ensures that the borrowing constraint $a \geq 0$ is satisfied. The growth-adjusted discount rate ρ in (9) is defined as $\rho = \tilde{\rho} - (1 - \gamma)g$.

The optimal consumption function $c_t(a, z)$ that solves the HJB is defined by

$$c_t(a, z) = [\partial_a V_t(a, z)]^{-\frac{1}{\gamma}}. \quad (10)$$

The associated savings function is denoted by

$$\varsigma_t(a, z) := r_t a + z - \tau_t(a, z) - c_t(a, z) \quad (11)$$

If a value function $V_t(a, z)$ solves the HJB (9) and satisfies the boundedness condition

$$\lim_{T \rightarrow \infty} \mathbb{E}_T [e^{-\rho T} V_T(a_{jT}, z_{jT})] = 0, \quad (12)$$

then the stochastic process for consumption defined by (10) solves the sequence version of the household problem (2).¹¹

The distribution of households across real asset holdings and endowment shares $g_t(a, z)$ satisfies the Kolmogorov Forward Equation (KFE)

$$\partial_t g_t(a, z) = -\partial_a [g_t(a, z) \varsigma_t(a, z)] - g_t(a, z) \sum_{z' \neq z} \lambda_{z, z'} + \sum_{z' \neq z} \lambda_{z', z} g_t(a, z'). \quad (13)$$

Let $f_t(\omega, z)$ denote the distribution of households across asset and endowment shares. For a given path of aggregate real wealth a_t , $f_t(\omega, z)$ and $g_t(a, z)$ are related by

$$f_t(\omega, z) = g_t(\omega a_t, z). \quad (14)$$

The KFE is a backward-looking equation where the initial distribution $g_0(a, z)$ is given.

2.2 Government

Nominal Government Budget Constraint. We assume a fiscal authority that issues short-term nominal government debt B_t subject to the budget constraint:

¹¹See Theorem 3.5.3 in [Pham \(2009\)](#). The expectation in (12) is with respect to the stochastic process for idiosyncratic income and assets for household j , given by the budget constraint (6).

$$dB_t = [i_t B_t - s_t P_t y_t] dt \quad (15)$$

where s_t is the ratio of primary surpluses to output and is determined by the tax and transfer function as

$$s_t = \int_{j \in [0,1]} \tau_t(z_{jt}) dj \quad (16)$$

Equation (15) defines the evolution of nominal government debt. This is a backward-looking equation where the initial level of nominal government $B_0 > 0$ is given. We restrict $B_t \geq 0$ so that the government can only borrow and not lend.¹²

De-trended Real Government Budget Constraint. We denote de-trended real government debt (or the debt-output ratio) by b_t ,

$$b_t = \frac{B_t}{P_t y_0 e^{gt}}. \quad (17)$$

For $t > 0$, real debt b_t evolves according to the real version of the government budget constraint given by (15):

$$db_t = [r_t b_t - s_t] dt. \quad (18)$$

Real debt increases whenever real interest rate payments exceed real primary surpluses. At $t = 0$, de-trended real debt b_0 is a jump variable given by the ratio of exogenously given initial nominal debt B_0 to the endogenous initial price level P_0 .

Fiscal Policy. For our baseline analysis we focus on a time-invariant tax and transfer function $\tau_t(z) = \tau^*(z)$, so that surpluses or deficits are a constant fraction of real output $s_t = s^*$. In Section 4.3, we generalize the analysis to allow for a broader class of fiscal rules of the form

$$s_t = s(b_t, r_t). \quad (19)$$

These rules allow primary surpluses to respond to real aggregate debt, real interest rates or real interest payments and play an important role in determining the price level when governments run persistent deficits, $s_t < 0$.

Monetary Policy. For our baseline analysis we focus on a nominal interest rate peg $i_t = i^*$. In our quantitative analysis in Section 5 we allow for long-term debt

¹²Introducing government consumption would be subsumed in s_t in Equation (15), thereby leaving the key mechanisms of our model unchanged.

and a richer class of Taylor-type rules for nominal interest rates. We also discuss how allowing for other monetary rules affects our results about the determination of the price level and inflation in Section 2.3.

2.3 Equilibrium

We first define a *real equilibrium* as a collection of real variables which satisfy household optimality, are consistent with their laws of motion, and obey market clearing.

Definition 1. *Given (i) a constant tax and transfer function $\tau^*(z)$; and (ii) an initial distribution of households across asset and endowment shares $f_0(\omega, z)$, a real equilibrium is a collection of variables:*

$$\{V_t(a, z), c_t(a, z), f_t(\omega, z), g_t(\omega a_t, z), a_t, b_t, r_t\}_{t \geq 0} \quad (20)$$

such that, for all $t \geq 0$:

1. the value function $V_t(a, z)$ solves the HJB (9) and satisfies the boundedness condition (12)
2. the consumption function is defined by (10)
3. the distribution of asset levels $g_t(\omega a_t, z)$ solve the KFE (13)
4. the distribution of household endowment shares $f_t(\omega, z)$ satisfies (14)
5. the path of government debt b_t satisfies the government budget constraint (18)
6. the asset market clears, $a_t = b_t$

Note that by Walras' law, asset market clearing implies that the goods market clearing condition is also satisfied:

$$\int_{j \in [0,1]} c_{jt} dj = 1 \text{ for all } t \geq 0.$$

Price Level and Inflation Determination. Under our assumptions about monetary and fiscal policy, each real equilibrium implies a unique initial price level P_0 and a subsequent unique path of inflation π_t . These are determined as follows. Each real equilibrium contains an initial value of real government debt b_0 . Since initial nominal debt B_0 is given, the initial price level is determined as

$$P_0 = \frac{B_0}{b_0}.$$

The path of inflation is uniquely determined by the equilibrium path of real rates r_t and the nominal rate i^* which is set by the monetary authority as

$$\pi_t = i^* - r_t - g.$$

It follows that uniqueness of a real equilibrium implies uniqueness of the price level. If there is more than one real equilibrium then there will be more than one possible path for the price level. But if the real equilibrium is unique, then there is only one possible path for the price level P_t for $t \geq 0$, which is determined by initial nominal debt and monetary policy. As a result, we focus most of our analysis on the existence and uniqueness of real equilibria, with the understanding that whenever the real equilibrium is unique, so too is the price level and inflation.

Monetary Policy Rules. With flexible prices, the equivalence between uniqueness of real equilibria and uniqueness of the path of prices does not depend on our assumption of a nominal interest rate peg $i_t = i^*$. If the monetary authority instead follows an instantaneous feedback Taylor Rule of the form

$$i_t = i^* + \phi_m(\pi_t - \pi^*) \tag{21}$$

then inflation is uniquely determined as

$$\pi_t = \frac{i^* - \phi_m \pi^* - r_t - g}{1 - \phi_m}.$$

If the monetary authority follows a lagged feedback Taylor Rule of the form

$$di_t = -\theta_m [i_t - i^* - \phi_m(\pi_t - \pi^*)] dt \tag{22}$$

then initial inflation is determined as $\pi_0 = i_0 - r_0 - g$ and subsequent inflation is determined as the unique forward solution to the ordinary differential equation

$$d\pi_t = -\theta_m [\pi_t - \phi_m(\pi_t - \pi^*) + r_t - (g - i^*)] dt - dr_t.$$

Depending on parameter configurations, prices and inflation may not remain bounded, but there is nothing in the equilibrium definition that rules out such paths.

3 Primary Surpluses $s^* > 0$

We start by showing uniqueness of equilibrium when the fiscal authority runs positive primary surpluses. We use an example to illustrate the different dynamics in the heterogeneous agent economy compared to its representative agent counterpart.

3.1 Stationary Equilibrium

Household Asset Demand. In a stationary equilibrium, the real rate r_t is constant. Under regularity conditions that are well understood, with a constant interest rate r and transfer function $\tau^*(z)$, the solution to (9) and (13) implies a unique stationary distribution $g(a, z; r)$.¹³ We use this result to construct a function $\mathbf{a}(r)$ that maps different interest rates into the aggregate quantity of real assets held by households in the corresponding stationary distribution,

$$\mathbf{a}(r) := \int_{a,z} ag(a, z; r)dadz$$

It is well known that $\lim_{r \rightarrow \rho} \mathbf{a}(r) = \infty$. In addition we will assume that the function $\mathbf{a}(r)$ is continuous, differentiable and strictly increasing.¹⁴ In Online Appendix C.2 we show that there exists an interest rate $\underline{r} < 0$ below which households do not hold any assets in the stationary distribution, so that $\mathbf{a}(r) = 0$ for all $r \leq \underline{r}$. The blue line in Figure 1 labelled $\mathbf{a}(r)$ is an example of a typical stationary asset demand function.

Government Asset Supply. In a stationary equilibrium, the government budget constraint defines a steady-state asset supply function $\mathbf{b}(r)$. This is obtained by setting $db_t = 0$ in (15),

$$\mathbf{b}(r) = \frac{s^*}{r}. \tag{23}$$

Since $b_t \geq 0$, this supply function takes the shape of a downward-sloping hyperbola in the positive quadrant as illustrated by the red line labelled $\mathbf{b}(r)$ in Figure 1.

Stationary Equilibrium. A stationary equilibrium requires that $\mathbf{a}(r) = \mathbf{b}(r)$, so that the asset market clears. Given our assumptions, there is a unique stationary real equilibrium shown as (b^*, r^*) in Figure 1. The assumption that primary surpluses are positive $s^* > 0$ implies that the stationary equilibrium real rate r^* is positive.

¹³See e.g. Bewley (1995), Stokey et al. (1989), and Aiyagari (1994).

¹⁴Achdou et al. (2022) show that sufficient conditions for this to be true are $\gamma \leq 1$ and $\underline{a} \geq 0$.

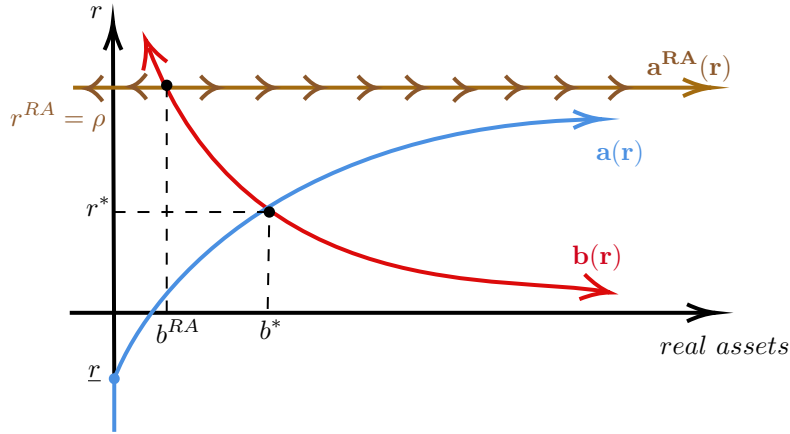


Figure 1: Steady state equilibrium with positive surpluses

The unique stationary equilibrium in the corresponding representative agent economy is the point (b^{RA}, r^{RA}) in Figure 1. In this economy the household asset demand curve is perfectly elastic at $r = \rho$. As is well known, in the heterogeneous agent economy the real rate is lower and the level of real government debt is higher than in the representative agent economy.

3.2 Non-Stationary Equilibrium

Because there is a unique stationary real equilibrium, in order to pin down the price level and inflation it suffices to rule out multiplicity of non-stationary real equilibria. Before tackling the heterogeneous agent economy, it is useful to recap the argument in the representative agent economy.

Uniqueness in Representative Agent Economies. In a representative agent economy, consumption satisfies an Euler equation of the form

$$\frac{dc_t}{c_t} = \frac{1}{\gamma} (r_t - \rho) dt$$

In an endowment economy, goods market clearing implies $dc_t = 0$ and hence in equilibrium $r_t = \rho$ at all points in time, not just in a stationary equilibrium. Graphically, this means that the economy lives on the brown horizontal line labelled $\mathbf{a}^{RA}(r)$ in Figure 1 at all points in time. The real government budget constraint implies that $db_t = [\rho b_t - s^*]dt$. It follows that real debt is increasing when it is above steady-state, and decreasing below steady-state, as illustrated by the arrows in Figure 1. Paths with increasing debt are ruled out as equilibria by showing that they violate a

household transversality condition. Paths in which debt is decreasing are ruled out since they violate the household's borrowing constraint in finite time. This argument is formalized in Online Appendix A. It follows that the stationary equilibrium is the unique real equilibrium and the initial price level and subsequent inflation are uniquely determined:

$$P_0 = \frac{B_0}{b^{RA}} \quad \text{and} \quad \pi_t^{RA} = i^* - \rho - g$$

Equilibrium paths display an initial jump in the price level at $t = 0$, and a constant inflation rate equal to steady-state inflation for $t > 0$.

Uniqueness in Representative Agent Economies with Bonds-In-Utility. The heterogeneous agent economy differs from the representative agent economy in part because the steady-state asset demand function is not perfectly elastic. In Online Appendix B we describe a simple representative agent economy in which households directly generate utility by holding real government debt. This economy features a steady-state asset demand function $\mathbf{a}^{BIU}(r)$ that has the same qualitative properties as $\mathbf{a}(r)$. In this economy, all equilibria lie on the one-dimensional manifold $\mathbf{a}^{BIU}(r)$ at all points in time, and away from steady-state the dynamics of government debt are unstable. A transversality condition and borrowing constraint rule out explosive paths in either direction as equilibria and hence the steady-state equilibrium is the unique equilibrium. The initial price level and subsequent inflation are uniquely determined. With positive primary surpluses, the difference between this economy and the standard representative agent economy is that the real interest rate is endogenous and depends on the level of surpluses. See Online Appendix B.3 for a formal argument.

State-Space Representation for Heterogeneous Agent Economy. Establishing that there is no multiplicity of non-stationary equilibria in the heterogeneous agent economy is more difficult than in the representative agent bonds-in-utility economy because the equilibria do not lie on a one-dimensional manifold. The aggregate state for the heterogeneous agent economy consists of the household asset and endowment distribution $g_t(a, z)$.¹⁵ It is useful to partition this distribution into two components,

¹⁵The absence of the interest rate r_t from the aggregate state is not immediately obvious. However, as we verify below, in equilibrium it is implied by the joint distribution $g_t(a, z)$. In our quantitative analysis, we consider unanticipated time-varying shocks to various exogenous parameters. In these

which we denote by $\Omega_t := \{f_t(\omega, z), b_t\}$

- (i) $f_t(\omega, z)$: the joint distribution of household asset shares and endowment shares
- (ii) b_t : the level of real government debt.

The reason for partitioning the aggregate state in this way is that the two components have different dynamic properties. The distribution $f_t(\omega, z)$ is backward-looking and cannot jump. The level of real debt is a jump variable. It can jump because different values of the initial price level P_0 revalue the outstanding stock of nominal bonds B_0 . Partitioning in this way makes it clear that although the household distribution $g_0(a, z)$ can jump, it can only jump along a single dimension such that the relative wealth holdings of each household remains unchanged. Using this state variable, we can write the consumption function $c_t(a, z)$ as $c(a, z, \Omega_t)$, where dependence on time is completely subsumed in the aggregate state.

Roadmap. Our discussion of uniqueness involves two steps. First, we show that any paths for b_t that diverge in either direction are not consistent with equilibrium because they involve eventual violation of either the borrowing constraint or a necessary household transversality condition. Second, we argue that the dynamics of Ω_t around the unique stationary equilibrium are locally saddle-path stable. Given an initial distribution $f_0(\omega, z)$ in the vicinity of $f^*(\omega, z)$, saddle-path stability implies that there is a unique initial value for the jump variable b_0 and unique subsequent paths of the aggregate state Ω_t such that the economy converges to $\Omega^* = \{f^*(\omega, z), b^*\}$.¹⁶

Ruling Out Explosive Equilibria. In Online Appendix C.3, we show that all paths of government debt b_t that grow at rate $r_t < \rho$ imply eventual violation of the following household transversality condition:

$$\lim_{T \rightarrow \infty} \mathbb{E}_{jt} [e^{-\rho T} c_T(a_{jT}, z_{jT})^{-\gamma} a_{jT}] \leq 0. \quad (24)$$

cases, the state space Ω_t needs to be expanded to include the law of motion for these exogenous driving processes.

¹⁶We must also rule out the possibility of non-stationary equilibria that remain bounded away from the stationary steady-state and involve cycles or similar dynamics. Although we cannot prove that no such equilibria exist, we have not encountered any numerically.

and hence cannot be part of equilibrium.¹⁷ Sufficient conditions for the equilibrium interest rate r_t in the heterogeneous agent economy to be below the discount rate ρ for all $t \geq 0$ are established in Not For Publication Appendix G.

Useful Characterization of Equilibrium Real Rate. In Online Appendix C.1 we derive expressions for expected consumption growth $\mathbb{E}_t [dc_{jt}]$ for constrained and unconstrained households. Here we use the short-hand notation $c_{jt} := c(a_{jt}, z_{jt}, \Omega_t)$ to denote the consumption of household j at time t . By aggregating these expressions across households, applying the law of iterated expectations, and imposing market clearing we derive the following relationship between the real rate and the aggregate state Ω_t ,

$$0 = \underbrace{\frac{\mathcal{C}_t^u}{\gamma}(r_t - \rho)}_{\text{intertemporal substitution}} + \underbrace{\frac{\mathcal{C}_t^u}{\gamma} \tilde{\mathbb{E}}_t^u \left[\sum_{z'} \lambda_{z_j, z'} \left(\frac{c(\omega_j, z', \Omega_t)}{c_{jt}} \right)^{-\gamma} \right]}_{\text{precautionary motive}} + \underbrace{\mathbb{E}_t \left[\sum_{z'} \lambda_{z_j, z'} \{c(\omega_j, z', \Omega_t) - c_{jt}\} \right]}_{\text{intertemporal smoothing}} \quad (25)$$

The expectation operator $\tilde{\mathbb{E}}_t^u$ is a consumption-weighted mean across the set of unconstrained households, and \mathcal{C}_t^u is the total consumption of unconstrained agents. Not For Publication Appendix F contains a full derivation of this relationship.¹⁸

Equation (25) can be interpreted as balancing three forces driving changes in aggregate consumption that must net out to zero in an endowment economy. The first term is an intertemporal substitution motive for saving. The second term is the average precautionary savings motive. The presence of \mathcal{C}_t^u captures the fact that this saving motive is only active for unconstrained households. The final term reflects an intertemporal motive for smoothing income shocks. In equilibrium, the interest rate is set so that the negative intertemporal substitution motive exactly offsets the combined effects of the precautionary saving and intertemporal smoothing motives.¹⁹

Equation (25) also confirms that the real rate is not required as a separate component of the aggregate state since that equation implicitly defines a time-invariant

¹⁷Establishing the transversality condition (24) as a necessary condition for household optimality is non-trivial. Kamihigashi (2001) shows that it is necessary in an analogous deterministic economy. Kamihigashi (2003) shows necessity in a discrete time stochastic economy.

¹⁸Not For Publication Appendix G contains the analogous formula for the real rate functional when idiosyncratic endowments follow a diffusion process.

¹⁹In the special case with quadratic utility, no borrowing constraints (hence, no precautionary saving) and $r_t = \rho$, equation (25) states that consumption is a martingale.

functional from Ω_t to r_t that holds at all times in equilibrium:

$$r_t = \mathbf{r} [\Omega_t]. \quad (26)$$

Local Saddle Path Stability. We derive the dynamics of the the aggregate state Ω_t by expressing the Kolmogorov Forward Equation (13) in terms of asset shares, and substituting the real rate functional (25) into the government budget constraint (18):

$$\begin{aligned} \partial_t f_t(\omega, z) &= -\partial_\omega \left[f_t(\omega, z) \frac{1}{b_t} \{z - \tau^*(z) - c(\omega b_t, z, \Omega_t) + s^* \omega\} \right] \\ &\quad - f_t(\omega, z) \sum_{z' \neq z} \lambda_{zz'} + \sum_{z' \neq z} \lambda_{z'z} f_t(\omega, z') \end{aligned} \quad (27)$$

$$\frac{db_t}{dt} = \mathbf{r} [\Omega_t] b_t - s^* \quad (28)$$

Since this system is comprised of a one-dimensional jump component b_t and an infinite dimensional backward looking component $f_t(\omega, z)$, local saddle-path stability requires that, around the steady-state, this PDE system has one positive eigenvalue and non-positive remaining eigenvalues.

Discretized Economy. Although we are not able to prove saddle-path stability for the full continuum economy, we have found the system to be saddle path stable in our numerical explorations of discretized versions of this economy. Here we offer some intuition for local saddle-path stability from this discretized economy.

We consider a discrete approximation to $f(\omega, z)$ on a grid for relative asset shares of size N_ω , which we denote by the $N \times 1$ vector \mathbf{f} where $N = N_\omega \times N_z$. In Online Appendix C.4 we show that the finite difference approximation the PDE system (27) is given by the system of $N + 1$ ODEs

$$\frac{d\mathbf{f}}{dt} = \mathbf{A}_\omega [\mathbf{f}_t, b_t]^T \mathbf{f}_t + \mathbf{A}_z^T \mathbf{f}_t \quad (29)$$

$$\frac{db_t}{dt} = \mathbf{r} [\mathbf{f}_t, b_t] b_t - s^* \quad (30)$$

The matrices $\mathbf{A}_\omega [\mathbf{f}_t, b_t]^T$ and \mathbf{A}_z^T are upwind finite difference approximations to the two linear operators that comprise the KFE for (ω, z) .²⁰

²⁰The transposes reflect the fact that these matrices are constructed by first constructing finite difference approximations to the adjoint operators in (27).

The dependence of $\mathbf{A}_\omega [\mathbf{f}_t, b_t]^T$ on the distribution \mathbf{f}_t and real debt b_t arises for three reasons. First, a change in aggregate wealth b_t has a common effect on the interest earnings at all points in the wealth distribution. This direct effect is reflected by the b_t in the denominator of the top line of (27). Second, a change in aggregate wealth impacts consumption of all households via a wealth effect. This is reflected in the b_t in the first argument of the consumption function in (27). Finally, there are further general equilibrium effects on consumption because of future interest rate dynamics. These are reflected in the dependence of the consumption function on the aggregate state Ω_t in the third argument.

In Online Appendix C.4, we linearize the discretized system (29) around the steady state (\mathbf{f}^*, b^*) and show that the local dynamics are approximately

$$\begin{pmatrix} \frac{d\mathbf{f}}{dt} \\ \frac{db}{dt} \end{pmatrix} \approx \begin{pmatrix} \mathbf{A}_\omega^{*T} + \mathbf{A}_z^T & \nabla_b \mathbf{A}_\omega^T [\mathbf{f}^*, b^*] \\ 0 & b^* \{ \partial_b \mathbf{r} [\mathbf{f}^*, b^*] - (-\frac{r^*}{b^*}) \} \end{pmatrix} \begin{pmatrix} \mathbf{f}_t - \mathbf{f}^* \\ b_t - b^* \end{pmatrix} \quad (31)$$

where term $\nabla_b \mathbf{A}_\omega^T [\mathbf{f}^*, b^*]$ is the $N_\omega \times 1$ vector of derivatives of \mathbf{A}_ω^{*T} with respect to real debt b .

The approximation in (31) refers to the zero in the bottom left element of the Jacobian. Our approximation requires this term to be small only relative to the term in the bottom right element of the Jacobian. This means we require that around the steady-state, the dynamics of real government debt are more sensitive to changes in the *level* of real debt, holding the distribution of asset shares constant, than to changes in the *distribution* of asset shares, holding the level of real debt constant.²¹

In this case, the Jacobian is approximately block triangular, allowing us to sign the eigenvalues of the full system: $\mathbf{A}_\omega^{*T} + \mathbf{A}_z^T$ is an irreducible transition rate matrix and so has a single zero eigenvalue and remaining negative eigenvalues. The sign of the remaining eigenvalue is given by the sign of $\partial_b \mathbf{r} [\mathbf{f}^*, b^*] b^* + \mathbf{r} [\mathbf{f}^*, b^*]$. The first term is the inverse of the derivative of the steady-state household asset demand curve, multiplied by the level of steady-state assets. The second term is the steady-state interest rate. As both terms are positive under constant positive surpluses, the remaining eigenvalue

²¹This assumption might appear at odds with our substantive messages that emphasize changes in the distribution of real wealth as a quantitatively important factor in driving inflation and price level dynamics. However, as our simulations confirm, these are not contradictory: the feedback from the distribution of shares to the debt dynamics are large enough to be quantitatively meaningful, but would need to be orders of magnitude larger to alter the qualitative features of the dynamic system.

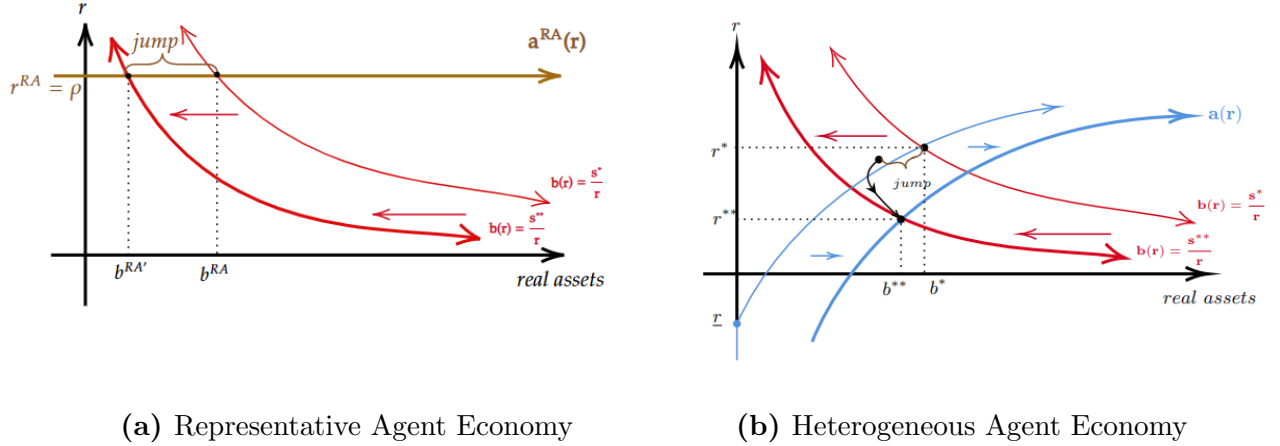


Figure 2: A permanent reduction in surpluses

is strictly positive and the economy saddle-path stable.

3.3 Example: Permanent Reduction in Surpluses

We use a permanent reduction in surpluses as an example to illustrate the saddle-path dynamics. Consider a fiscal authority that unexpectedly changes the tax function from $\tau^*(z)$ to $\tau^{**}(z) = (1 - \Delta_s)\tau^*(z)$ so that primary surpluses decline to $s^{**} = (1 - \Delta_s)s^*$, with $\Delta_s \in (0, 1)$. The new steady-state government bond supply function is $\mathbf{b}(r) = \frac{s^{**}}{r}$, which is displayed as a leftward shift of the red line in Figure 2.

First, consider the effects of this change in the representative agent economy. The initial steady-state equilibrium before the change is indicated by b^{RA} . When the level of surpluses fall, the economy immediately jumps to the new steady-state equilibrium at the point labelled $b^{RA'}$. The level of real debt immediately falls to $(1 - \Delta_s)b^{RA}$, which is achieved by a one-time upward jump in the price level from P_0 to $\frac{P_0}{1 - \Delta_s}$ with no change in either the real interest rate or inflation. The stock of nominal debt is unchanged, but real surpluses are reduced and thus the price level must jump to lower the real value of outstanding debt.

In the heterogeneous agent economy, the initial steady-state equilibrium is indicated by the point (b^*, r^*) . In contrast to the RA model, a change in the tax and transfer function induces a shift in the steady-state household asset demand function for two reasons: (i) it affects disposable income; and (ii) it alters the degree of risk-sharing in the economy. In this example, the effect is to shift the $\mathbf{a}(r)$ curve to the right. The new steady-state after the change is indicated by the point (b^{**}, r^{**}) .

Unlike in the representative agent economy, the economy does not jump immediately to the new steady-state. Rather, saddle-path dynamics imply that on impact of the change there is a one-time jump in the level of real debt to the unique value of b_0 that is consistent with non-explosive dynamics, which then determines a unique r_0 through the real rate functional (25). This is indicated by the leftward jump in Figure 2b. The initial jump is achieved by a rise in the price level that devalues all households' wealth proportionately.²² This shift in the wealth distribution then induces trading among households as the interest rate falls smoothly to its new steady-state level. Without any change in monetary policy, inflation rises smoothly during this transition until it reaches its new steady-state level, which is higher than in the original steady state by the amount $r^{**} - r^*$.

4 Primary Deficits $s^* < 0$

We now assume that the fiscal authority runs a constant primary deficit. We first show that there are zero or two steady-state equilibria, depending on the level of deficits. We then characterize the out of steady-state dynamics and non-stationary real equilibria. We end this section with a discussion of alternative ways to restore uniqueness of a saddle-path stable equilibrium and hence a unique path for prices.²³

4.1 Stationary Equilibria

The household asset demand $\mathbf{a}(r)$ function is qualitatively unchanged with $s^* < 0$. However, the steady-states of the government budget constraint, $\mathbf{b}(r) = s^*/r$ is an upward-sloping hyperbola for $b_t \geq 0$, as depicted in Figure 3. Note that with $s^* < 0$, any steady-state equilibria must have a real rate that is below the growth rate of the economy $r^* < 0$. From Figure 3, it is immediate that if such a steady-state equilibrium exists, then generically there will be two steady-state equilibria, as indicated by the two intersections of the asset supply and demand curves.²⁴ For a given nominal

²²In general, the initial jump in the price level may undershoot or overshoot its long-run value depending on the nature of the transfer function.

²³In Online Appendix C.5 we consider the case where $s^*=0$. Like in the case with $s^* > 0$, there is a unique equilibrium with a finite price level and the path of prices is uniquely determined. The steady-state real interest rate and real assets are $r^* = 0$ and $b^* = \mathbf{a}(0)$, respectively.

²⁴This conclusion follows from the existence of a \underline{r} such that for all $r_t < \underline{r}$, households do not save, meaning that the household steady-state asset demand curve intersects the $b = 0$ axis at a finite

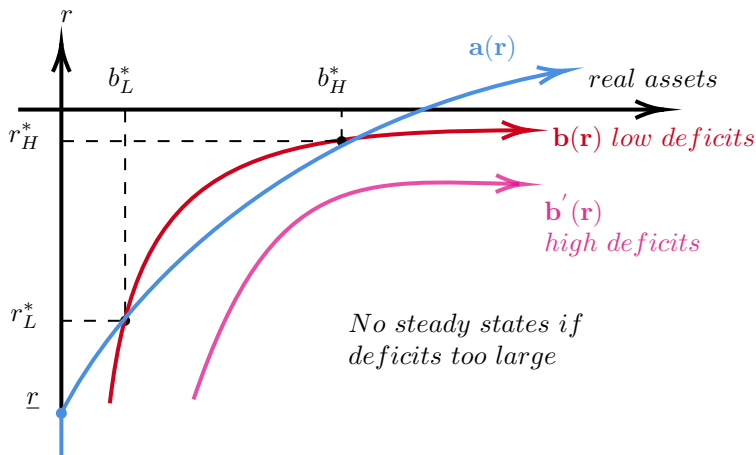


Figure 3: Maximum steady-state deficits

interest rate, the top equilibrium (b_H^*, r_H^*) has a higher level of real debt, higher real interest rate and lower inflation than the bottom equilibrium (b_L^*, r_L^*) . In Online Appendix C.6 we show that the high interest rate steady-state Pareto dominates the low interest rate one by reducing the volatility of individual consumption growth.

Maximum Deficits. There exists a maximum level of deficits that is consistent with the existence of a stationary equilibrium where the price level is finite and government debt is valued. As the level of deficits increases, the government asset supply curve shifts downward to the right, as illustrated in Figure 3. The maximum deficit is attained when the asset supply and demand curves are tangent to each other, which occurs at the point where the interest-rate elasticity of the steady-state household asset demand curve is equal to unity: $\mathbf{a}'(r)r/\mathbf{a}(r) = -1$.

This condition reflects the fact that the maximum attainable level of deficits depends on the strength of households' desire to hold assets for precautionary reasons. It follows that a change in the nature of after-tax idiosyncratic endowment risk can shift the asset demand curve $\mathbf{a}(r)$ and hence the maximum deficit. Any reduction in s^* must be implemented via a change in the function $\tau^*(z)$. Depending on the change in progressivity, the maximum deficit may increase or decrease through a shift in $\mathbf{a}(r)$. In general, a change in the tax function that reduces the amount of uninsured risk will lower the maximum attainable deficit because households have less incentive to accumulate precautionary savings.²⁵ In Section 5 we use our calibrated model to

interest rate \underline{r} . This discussion maintains the assumptions outlined in Section 3.1 so that $\mathbf{a}(r)$ is monotonically increasing. Without these assumptions, there is generically an even number of steady states.

²⁵Amol and Luttmer (2022) also emphasize that fiscal space depends on the overall level of risk

illustrate these forces.

Non-uniqueness of Price Level and Inflation. Since there are two steady-state equilibria with $s^* < 0$, standard FTPL arguments for uniqueness of the price level do not hold. Additional assumptions on fiscal policy must be imposed, or other modifications made to the economy, in order to uniquely pin down the price level and inflation. We discuss these possibilities in Section 4.3, but first we characterize the set of non-stationary equilibria.

4.2 Non-stationary Equilibria

Local Dynamics. We can characterize the local dynamics around each of the two steady states following the same line of argument as we did for the case with $s^* > 0$. The dynamics obey the same PDE system (27). The arguments we gave for why the eigenvalues associated with the backward looking component $f(\omega, z)$ are all non-negative remain unchanged. As before, we sign the eigenvalue associated with the jump variable b_t by assuming that – in the vicinity of a steady-state equilibrium – the effect on government debt dynamics due to general equilibrium feedback from movements in the distribution are small relative to the overall effect of changes in interest payments:

$$\frac{db_t}{dt} \approx b^* \left\{ \partial_{b\mathbf{r}} [f^*, b^*] - \left(\frac{r^*}{b^*} \right) \right\} \quad (32)$$

The term in braces is the difference between the slopes of the steady-state asset demand function ($\partial_{b\mathbf{r}}[\Omega^*] = (\partial_r \mathbf{a}[r^*])^{-1}$) and the steady-state bond supply function ($-\frac{r^*}{b^*} = (\partial_r \mathbf{b}[r^*])^{-1}$). The eigenvalue associated with government debt b_t is therefore positive at the top steady-state, where the asset demand function crosses the asset supply function from below, and is negative at the bottom steady-state, where the asset demand function crosses the asset supply function from above. Hence the local dynamics around the top steady-state are saddle-path stable, similarly to the unique steady-state in the case with surpluses. The dynamics around the bottom steady-state are locally stable. Simulations confirm these properties.

Figure 4 illustrates these dynamics. For a given initial distribution $f_0(\omega, z) \neq f^*(\omega, z)$, there is a unique equilibrium converging to (b_H^*, r_H^*) and a continuum of equilibria converging to (b_L^*, r_L^*) , indexed by the initial level of real debt b_0 . Con-

in an economy in which households face idiosyncratic shocks to their returns on capital.

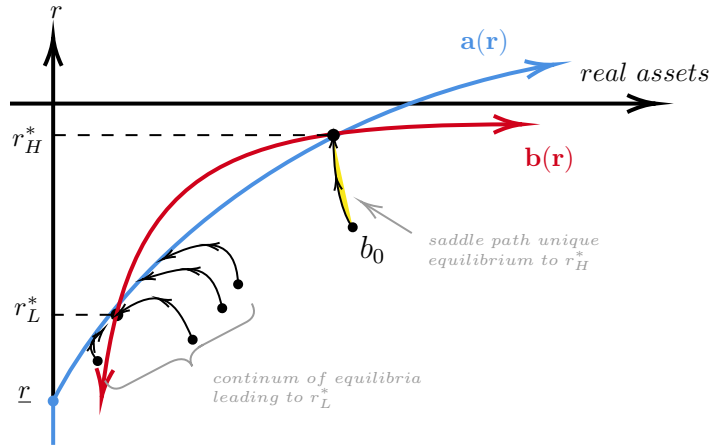


Figure 4: Non-stationary equilibria with deficits. For a given $f_0(\omega, z) \neq f^*(\omega, z)$, there are a continuum of equilibria indexed by initial real government debt.

quently, the price level and inflation are not pinned down without additional assumptions that rule out almost all of these equilibria. Because it is the top equilibrium that is saddle-path stable, there is a lower bound on the initial price level that is consistent with equilibrium. This minimum initial price level is given by $P_0 = \frac{B_0}{b_0}$, where b_0 is the unique initial value of real debt for which the economy converges to the top saddle-path stable equilibrium.

Exact Characterization in a Bonds-In-Utility Economy. In Online Appendix B.4 we show that the representative agent economy with bonds in the utility function has qualitative steady-state properties that are the same as in the heterogeneous agent economy. In that economy, $\mathbf{a}^{BIU}[r]$ can be derived in closed form, and we can fully characterize the global dynamics: the top steady-state is unstable, the bottom steady-state is stable and there is a lower bound on the initial price level.

4.3 Options for Price Level Determination

Multiplicity of equilibria poses a challenge for quantitative work. We show that there are several ways to eliminate the locally stable steady-state and achieve uniqueness. First, through certain fiscal policy rules. Second, by introducing a foreign sector with relatively inelastic demand for domestic government debt. Lastly, through a form of long-run inflation anchoring.

Real Debt Reaction Rule. Until now we have assumed a fiscal rule that keeps primary deficits constant. Assume instead that the fiscal authority follows a rule in

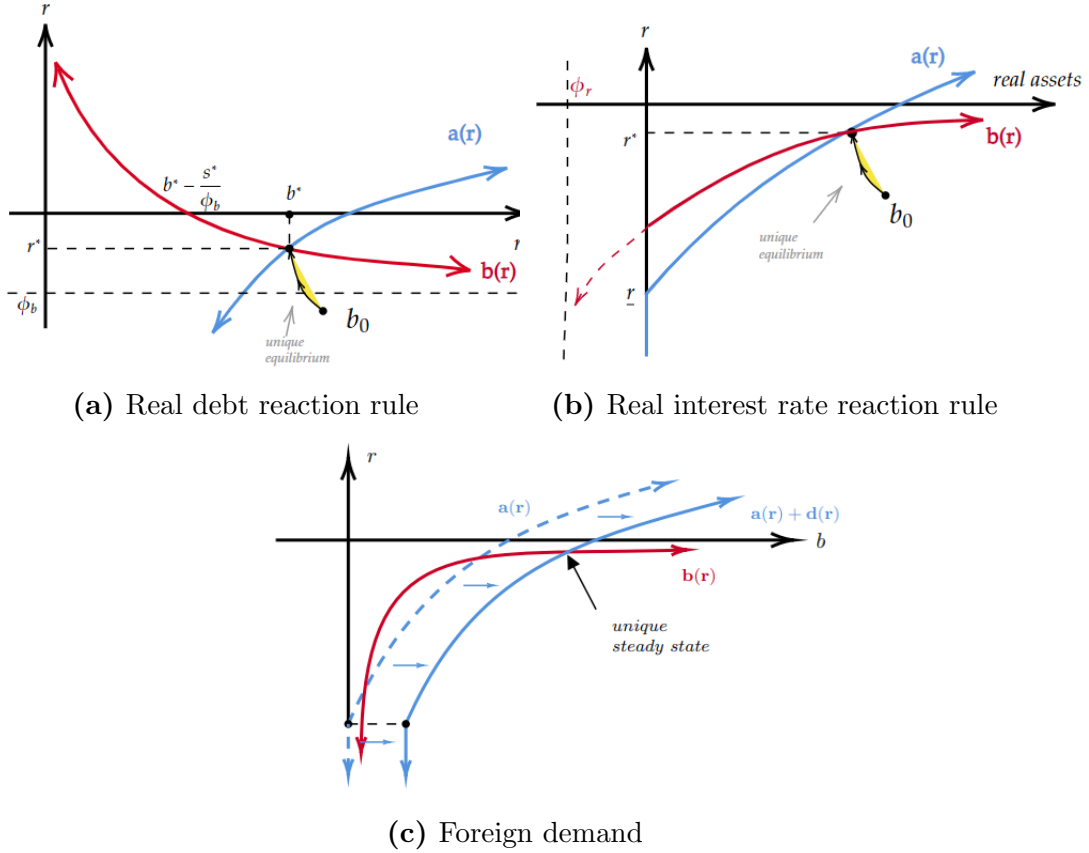


Figure 5: Alternative approaches to deliver a unique equilibrium with deficits which primary deficits respond to real debt deviations from the steady-state level b^* :

$$s_t = s^* + \phi_b (b_t - b^*). \quad (33)$$

The steady-state level of deficits is denoted by $s^* < 0$. Outside of steady-state, the fiscal authority varies deficits by changing the tax and transfer function $\tau_t(z)$. The steady-state government asset supply curve is given by $r = \phi_b + \frac{s^* - \phi_b b^*}{b}$. If $\phi_b < r^* < 0$, then for $b > 0$, this is a downward sloping curve that intersects the household asset demand curve only once, as illustrated in Figure 5a.²⁶ There exists a unique steady-state equilibrium which is saddle-path stable and hence the initial price level and subsequent inflation are uniquely determined. Online Appendix C.7 contains details. Note that the condition $\phi_b < r^*$ implies that when outstanding

²⁶The household asset demand curve will also be affected, since higher levels of debt are associated with different transfer functions, which may alter the shape of the asset demand curve. In practice this effect can be made small by changing the level of deficits in an approximately distributional neutral way.

debt falls below its steady-state level, the government responds by cutting primary deficits. This reaction has a destabilizing effect on the debt accumulation process, which eliminates the bottom (stable) steady-state (b_L^*, r_H^*) .²⁷

Real Rate Reaction Rule. An alternative fiscal rule that also eliminates the stable steady-state equilibria is one in which primary deficits respond to deviations of the equilibrium real rate from its steady state,

$$s_t = s^* + \phi_r (r_t - r^*). \quad (34)$$

In Online Appendix C.8, we show that a sufficient condition to eliminate the stable steady-state is $\phi_r < \frac{s^*}{r^* - \mathbf{a}^{-1}(0)} < 0$. Figure 5b illustrates this case. When the real rate falls below its steady-state value, the fiscal authority cuts primary deficits. This response has a destabilizing effect that eliminates the bottom stable steady-state.

Interest Payment Reaction Rule. We also consider a fiscal rule in which primary surpluses respond to deviations of real interest payments from their steady state level:

$$s_t = s^* + \phi_s (r_t b_t - s^*). \quad (35)$$

In Online Appendix C.9 we show that the steady-state equilibria are unchanged from the baseline ($\phi_s = 0$). With an “active” rule ($\phi_s < 1$), the stability properties of the two steady-states are also unchanged. However, with a “passive” fiscal rule ($\phi_s > 1$), the stability properties of the two steady-states are reversed: the top steady-state is locally stable and the bottom one is saddle-path stable.

Inelastic Foreign Demand. If there is additional demand for government debt that is sufficiently interest-inelastic, for example from a foreign sector, then the bottom steady-state can be eliminated and uniqueness restored.

Denote the foreign demand for government debt as a function of the domestic real interest rate as $\mathbf{d}(r)$. The asset market clearing condition becomes $\mathbf{a}(r) + \mathbf{d}(r) = \mathbf{b}(r)$. To clearly see the effect of additional foreign demand, assume that it is perfectly

²⁷This rule has the somewhat unappealing feature that when government debt rises above its steady-state level, the government responds by running even larger primary deficits. However, this property is not important for uniqueness; the role of the rule is to eliminate the stable equilibrium with *low* levels of government debt. Upward explosive dynamics are ruled out even with a constant deficit policy as explained in Section 3. For example an asymmetric policy, in which primary deficits respond only to reductions in government debt would suffice for uniqueness.

inelastic, so that $\mathbf{d}(r) = b^f$. The overall asset demand curve is shifted to the right and the bottom steady-state disappears, as illustrated in Figure 5c. In Online Appendix D we offer a microfoundation based on a representative agent foreign sector that has bonds-in-utility preferences. We show that an interest rate elasticity of demand below one is sufficient to ensure that the two curves intersect only once.

Long-Run Inflation Anchoring. The previous approaches to delivering a unique path of prices work by making assumptions that eliminate the high inflation stable steady-state, leaving only the low inflation saddle-path stable steady state. An alternative route to uniqueness is to instead eliminate all dynamic equilibria that lead to the high inflation steady-state, leaving only the unique equilibrium leading to the low inflation steady-state. In Not For Publication Appendix H, we show that a central bank that coordinates *long-run* inflation expectations can successfully pin down the inflation and the price level in the *short-run* under a constant deficit fiscal policy rule.

5 Quantitative Exercises with Persistent Deficits

In this section we describe various quantitative experiments for a calibrated version of the model with persistent deficits in order to illustrate the role of redistribution and precautionary saving in shaping price level dynamics.²⁸

5.1 Model Extensions

We incorporate the following two extensions of the baseline model.

Extension I: Unsecured Household Credit. We allow for a non-zero borrowing limit. This permits nominal positions to be negative, thereby allowing some households to experience a positive wealth effect from an unanticipated rise in the price level, as in Doepke and Schneider (2006) and Auclert (2019). We assume that households can borrow up to a fixed limit that is denominated in real terms. We interpret it as unsecured borrowing, such as credit card debt, and impose an exogenous wedge between borrowing and saving rates. See Online Appendix E.1 for details.

²⁸Our economy is a flexible price, endowment economy in continuous time. In reality, the price level does not jump. Rather, the initial bursts of inflation from these shocks are drawn out over a period of time. Despite this simplification, the general forces at work are informative about the two-way feedback between the equilibrium wealth distribution and movements in the price level.

Parameter	Value	Target
Preferences		
γ Inverse EIS	1	
ρ Discount rate	2.8% p.a.	debt-to-annual GDP ratio of 1.10
Income Process		
g Real output growth	2.0% p.a.	average growth rate post-war
λ Arrival rate of earnings shocks	1.0 p.a.	
σ St. Dev. of log quarterly earnings	1.2	
Household Borrowing		
a Borrowing limit	\$15,000	70% of quarterly household earnings
$r^b - r$ Borrowing wedge	16% p.a.	average rate on credit card debt
Tax and Transfers: $\tau(z) = \tau_0 - \tau_1 * z$		
τ_1 Proportional tax rate	30%	personal taxes / labor income
τ_0 Lump sum transfer	33.3% of GDP	deficit: $s^* = -3.3\%$
Government Debt		
δ Maturity rate of government debt	20% p.a.	average duration of 5 years
Monetary Policy		
i Nominal rate	1.5%	average Federal Fund Rate

Table 1: Calibrated parameter values and targets.

Extension II: Long-Term Debt. We assume that the government issues long-term debt with a constant maturity rate. The switch to long-term debt has no effect on the preceding analysis of price level determination. However, as shown by Sims (2011) and Cochrane (2018), debt duration plays a key role in the dynamics of inflation after unanticipated changes in the nominal interest rate. This mechanism surfaces in some of our experiments where we explore monetary policy rules beyond an interest rate peg. Online Appendix E.2 describes the model with long-term debt.

5.2 Parameterization

Preferences. We set the elasticity of inter-temporal substitution γ to 1 so that households have log utility. We choose the discount rate ρ to match an annual debt-to-GDP ratio of 1.10 in the low inflation steady state. This target, which corresponds to the debt-to-GDP ratio in US data for the years leading up to the pandemic (2014-2019), implies a calibrated annual discount rate of 2.8%.

Endowment Process. We assume an annual aggregate real growth rate g of 2%, which was the US per-capita average over the post-war period.²⁹ Idiosyncratic endowment shares follow an $N_z = 5$ state process, with switching rates chosen so that income shocks arrive on average once per year and the endowment process generates a standard deviation of log quarterly earnings of 1.08, in line with US micro data.³⁰

Household Borrowing. We set the borrowing limit \underline{a} to \$15,000, which is approximately 70% of average quarterly household earnings to match the median credit card limit for working-age population in the Survey of Consumer Finances (SCF) (Kaplan and Violante, 2014). We set the wedge between the interest rates on borrowing and saving to 16% p.a., based on typical interest rates on unsecured credit card debt.³¹ Because of this exogenous wedge, the real borrowing rate is positive, and the natural borrowing limit is finite and exceeds the ad-hoc limit.

Tax and Transfer System. The tax and transfer system consists of a lump-sum transfer and proportional tax,

$$\tau(z) = -\tau_0 + \tau_1 z.$$

We set the proportional tax rate τ_1 to 30% to match the ratio of personal taxes and social insurance contributions to total labor income (NIPA Table 2.9) for 2014-2019. We then set the lump-sum transfer τ_0 at 33.3% of aggregate output to generate a primary deficit s^* of -3.3% of GDP, the average for the US over that period.³²

Government Debt. We assume that 20% of outstanding government debt matures each year to match a weighted average duration of 5 years (US Treasury). Given our target debt-to-GDP ratio of 110%, and primary deficit of 3.3%, the implied steady-state real interest rate equals $\frac{s^*}{B^*} + g = \frac{-0.033}{1.1} + 0.02 = -1\%$ p.a.

Monetary Policy. We assume that the central bank pegs the nominal rate at 1.5% p.a., consistent with the average interest rate target in the years leading up to the

²⁹See Series A939RX0Q048SBEA_PC1 from FRED, Federal Reserve Bank of St. Louis, <https://fred.stlouisfed.org>.

³⁰See, for example, the Global Repository of Income Dynamics (GRID), <https://www.grid-database.org/>.

³¹See Table Consumer Credit - G19, Federal Reserve Board, <https://www.federalreserve.gov/releases/g19/current/>.

³²The data sources for debt and deficits are series GFDEGDQ188S and FYFSGDA188S from FRED.

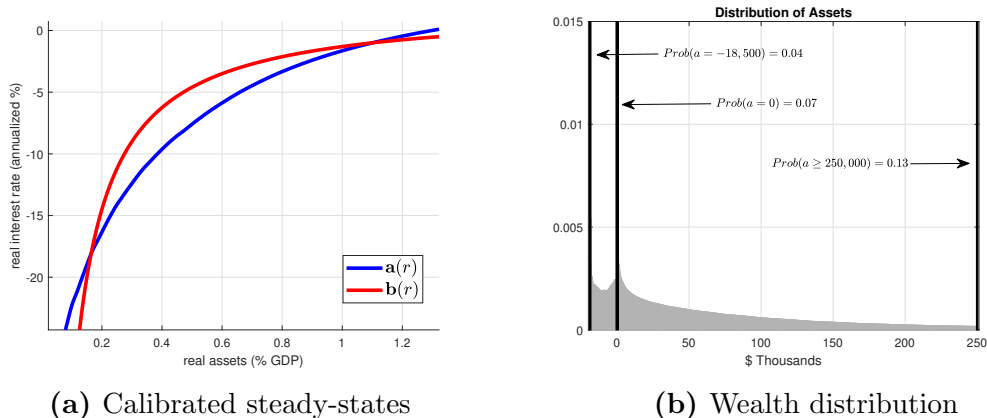


Figure 6: Calibrated steady-states and wealth distribution

pandemic. With a real interest rate of -1% , the implied annual inflation rate is 2.5% .

5.3 Properties of Steady States

Figure 6a displays the two stationary equilibria implied by our calibration. In line with our targets, the low inflation saddle-path stable steady-state has an annual debt-to-GDP ratio of 110% and an annual inflation rate of 2.5% . The high inflation steady-state has an annual debt-to-GDP ratio of 17.5% , and an annual inflation rate of around 19.5% . In what follows, we focus on the low-inflation steady state.

Wealth and MPC Distribution. Figure 6b and Table 2 illustrate that the model is broadly consistent with the distribution of liquid wealth in the 2019 SCF.³³ Expressed in 2019 dollars, mean and median household wealth in the model are $\$116,000$ and $\$40,000$ respectively. 19% of households have negative wealth and 27% of households have less than $\$1,000$. These moments were not targeted in our calibration, which was disciplined by aggregate statistics on national debt.

The average quarterly MPC in the model is around 14% , with the highest MPCs among the low-income households that either have close to zero wealth and so are near a kink in their budget constraint, or have substantial negative wealth and so are close to the borrowing limit.³⁴

³³Our definition of liquid wealth includes money market, checkings, savings, and call accounts, as well as directly held mutual funds, stocks and bonds, minus credit card and uncollateralized debt. We exclude the top 1% of households in the SCF by liquid wealth because of the well-known difficulties in matching the right-tail of the wealth distribution in this class of models.

³⁴Not For Publication Appendix J contains additional details on the distributions of wealth and marginal propensities to consume in the model.

Table 2

Mean liquid assets	Data	Model
Mean assets	\$116,000	\$100,317
Frac. with $a < \$0$	20.67%	19%
Frac. with $a < \$1,000$	37%	27%

Note: Moments of the wealth distribution in the model and the data. Monetary values expressed in 2019 dollars. Data is from the 2019 Survey of Consumer Finances (SCF) with the top 1% of households by liquid wealth are excluded. See the main text for the definition of liquid assets in the data.

Maximum Sustainable Deficit. As discussed in Section 4.1, there exists a maximum possible level of permanent deficits consistent with existence of an equilibrium where debt is valued. The size of this maximum deficit depends on whether it is reached by expanding lump-sum transfers or cutting proportional taxes. Under our calibration, raising transfers yields a maximum deficit of 4.6% of output, a 39% increase from the baseline steady-state value of 3.3%. Instead, lowering taxes allows the government to run a maximum deficit of 4.8%, a 45% increase from the baseline.

Lower proportional tax rates are, in general, associated with higher maximum steady-state deficits because they increase the volatility of disposable earnings. Households therefore bear more uninsured idiosyncratic income risk which raises their overall precautionary demand for safe liquid assets. For a given interest rate r , households are willing to hold more government bonds if they bear more idiosyncratic risk, giving the government more room to expand its deficit. Graphically, a lower value for τ_1 induces an outward shift in the the steady-state household asset demand curve (recall Figure 3). The same logic, with signs reversed, applies to an expansion of lump-sum transfers because they reduce the volatility of net earnings.

The role of precautionary saving is quantitatively important. For example, in an extreme case without proportional taxes ($\tau_1 = 0\%$), the maximum sustainable deficit that can be achieved by expanding transfers is 9.5%, almost three times as large as in our baseline. For similar reasons, when households are prohibited from borrowing, the maximum sustainable deficit rises to 5.9%. A key lesson from these experiments is that reforms that loosen credit, make tax and transfer systems more progressive, or provide more insurance to households reduce future fiscal space available to the government. These reforms restrict the government’s ability to expand deficits or cut surpluses, and therefore may constrain its ability to use expansionary fiscal policy to respond to adverse aggregate shocks.

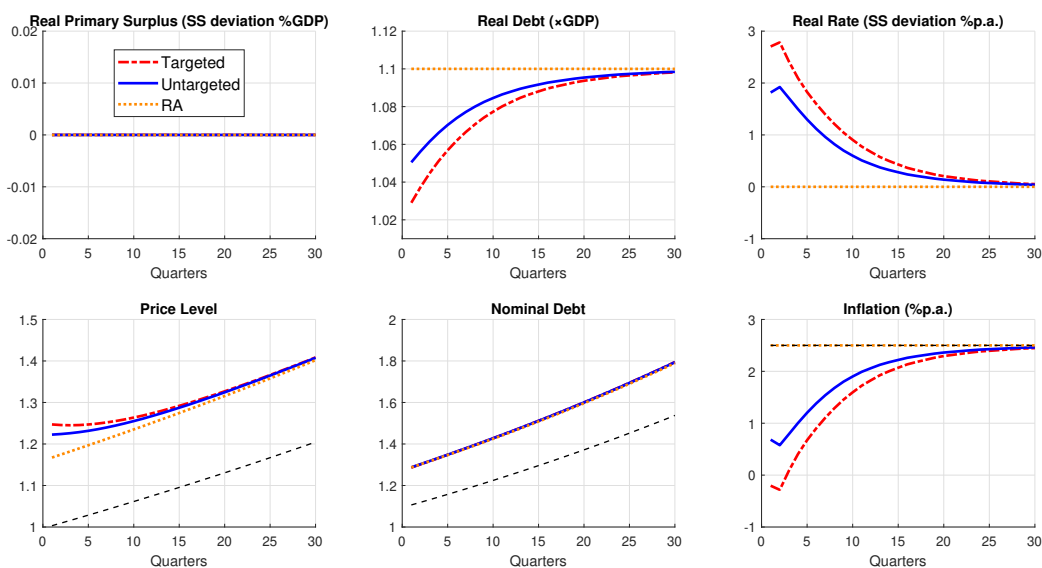
Implications for Secular Stagnation. A recent literature argues that the secular decline of real rates observed in the US and other developed economies is due to rising income risk and inequality, which has been accelerated by the sharp debt deleveraging that occurred after the 2008 financial crisis (Auclert and Rognlie, 2018; Eggertsson et al., 2019; Mian et al., 2021b). The argument is that higher inequality leads to a redistribution of income from the high-MPC poor to the low-MPC rich, which increases overall demand for wealth in the household sector. Similarly, more uninsured income risk or a tighter borrowing limit create a stronger precautionary motive, which increases demand for government bonds. These forces all manifest as an outward shift of the household asset demand function $\mathbf{a}(r)$. In a conventional economy with positive rates and permanent surpluses, such outward shifts in household asset demand indeed leads to a lower steady-state real rate.

However, in an economy with permanent deficits and a negative real rate, these comparative statics are reversed when the economy starts in the low-inflation steady state. An outward shift of the household asset demand function $\mathbf{a}(r)$ leads to a higher steady-state real rate. The reason is that in order to finance the same level of deficits with a higher quantity of debt, a less negative (i.e. higher) real rate is needed. This observation adds an important qualification to the commonly held view that shifts in the income distribution, income risk or deleveraging are candidate explanations for secular stagnation. In Section 5.5, we propose an alternative explanation for secular stagnation, rooted in the observation that in heterogeneous agent economies with persistent deficits and $r < g$, larger primary deficits depress the real rate.

5.4 Fiscal Helicopter Drop

Our first experiment is inspired by the experience of the US and other developed countries in the wake of the COVID-19 shock. In response to the disruptions caused by the pandemic, the US issued a large quantity of additional government debt and distributed much of the proceeds to households. We capture the core features of this fiscal helicopter drop by simulating an unexpected one-time issuance of nominal debt equal to 16% of initial outstanding government liabilities (equivalent to the observed 16% rise in the US debt-GDP ratio in 2020), which is distributed as a one-time lump-sum transfer to households. We consider two versions of this policy: one where transfers are distributed uniformly and one where transfers are distributed only to

Figure 7



Note: This figure plots impulse responses to a targeted and untargeted helicopter drop, aggregated at the quarterly frequency. The helicopter drop is a one-time issuance of 16% of total government nominal debt outstanding at $t = 0$. Only households in the bottom 60% of the wealth distribution receive the issuance in the targeted experiment (dashed red line). The orange line plots dynamics in the representative agent (RA) model. The dashed black line plots the initial steady state.

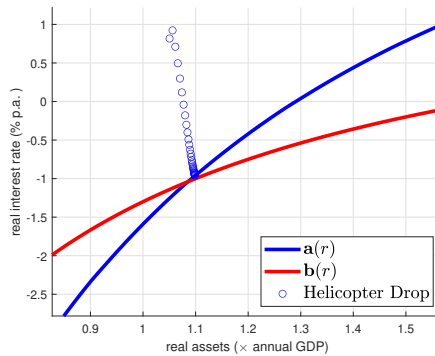
households in the bottom 60% of the wealth distribution, in line with the actual US experience.

Aggregate Effect of Fiscal Helicopter Drop. The effects of the fiscal helicopter drop are displayed in Figure 7. Since there are no changes to primary surpluses or any other structural parameters, the helicopter drop has no permanent real effects: the household and government nullclines are unchanged, and the economy converges back to its initial steady-state.

In the representative agent version of this economy, which is shown by the orange dotted line labelled “RA” in Figure 7, convergence is instantaneous.³⁵ The jump in the price level exactly offsets the new issuance of nominal debt so that the level of

³⁵The representative agent economy is constructed to have the same steady-state debt-to-GDP ratio as in the heterogeneous agent economy. However, since the representative agent economy does not admit a steady-state with persistent deficits, we assume an annual surplus-to-GDP ratio of 3.3% and an equilibrium real rate of 1%. We adjust the nominal interest rate so that the inflation rate is the same in the two economies.

Figure 8



Note: This figure shows the computed saddle-path dynamics from a one-time issuance of nominal government debt in (r_t, b_t) space. The total issuance amounts to 16% of nominal government debt outstanding at $t = 0$. The blue dots depict quarterly aggregates.

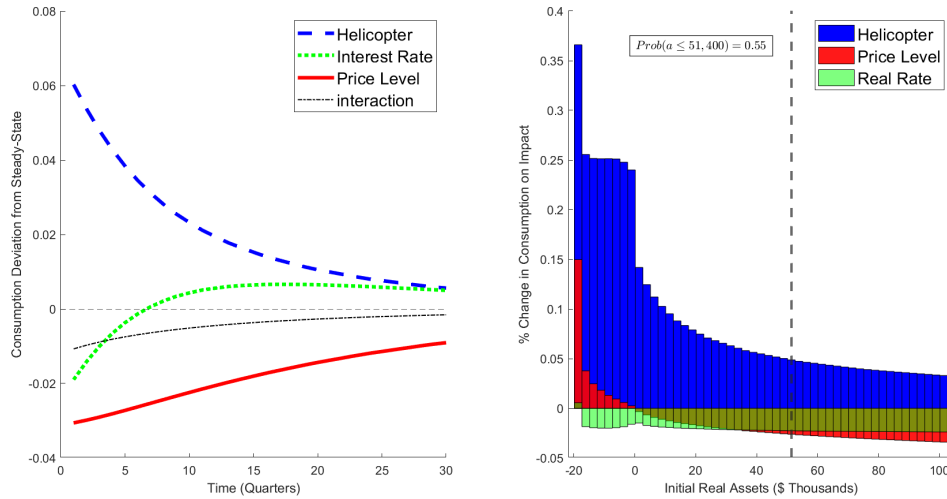
real debt remains constant and there are no further effects of the shocks.³⁶ However, in the heterogeneous-agent model, there are transitional dynamics. The computed saddle-path dynamics associated with this convergence in (r_t, b_t) space are displayed in Figure 8. The initial jump in the price level (bottom-left panel of Figure 7) is about 21%, higher than in the representative agent model, which more than offsets the 16% rise in nominal debt.

Why does an identical expansion in government debt place more upward pressure on the price level in the heterogeneous agent economy? The fiscal helicopter drop entails a redistribution of real wealth from high- to low-wealth households because the lump-sum transfer is progressive. Since the average MPC is higher among low wealth households, this redistribution raises the economy-wide desire to consume. With a constant aggregate endowment, the real interest rate must rise to restore goods market clearing. The higher (i.e. less negative) real interest payments require a reduction in total real government debt outstanding. Since nominal debt is fixed after the helicopter drop, the price level must then increase further. An alternative interpretation is simply that the additional spending pressure from redistribution, beyond the aggregate wealth effect, places more upward pressure on nominal prices than in the representative agent economy where only the wealth effect is present.

Decomposition of Fiscal Helicopter Drop. In addition to the the direct redistributive impact of the fiscal helicopter drop, there are two additional indirect

³⁶The initial price jump in Figure 7 is slightly more than 16% because in this and other figures, we plot impulse response functions aggregated to a quarterly frequency.

Figure 9



Note: This figure decomposes the effect of the helicopter drop on consumption into its general equilibrium sub-components. The left panel depicts how each sub-component affects aggregate consumption over time in isolation. The right panel depicts the effect of each sub-component on initial consumption across the wealth distribution. The dashed black line on the right panel delineates households that experienced initial consumption gains and losses as a result of the helicopter drop in 2019 US dollars.

general equilibrium channels at play that shape the subsequent dynamics of the real rate and inflation. First, the upward jump in the price level redistributes wealth from savers to borrowers, and dilutes the real savings for households with a positive net nominal position. Second, the resulting rise in the real rate leads households to postpone consumption. The left panel in Figure 9 displays the dynamic effects of each of these channels on aggregate consumption. The helicopter drop itself raises consumption, while the higher price level lowers consumption. These effects diminish as the economy returns to steady-state. The higher real interest rate leads households to delay consumption, which is reflected by the initially lower but subsequently higher consumption in the green dotted line in Figure 9.

The aggregate decomposition masks substantial heterogeneity in the effect of these channels across households. The right panel of Figure 9 shows the contribution of each channel to the change in consumption on impact along the wealth distribution. Low-wealth households increase consumption substantially, predominantly due to their higher MPCs out of the direct helicopter drop at the steady-state price level. In addition, the jump in the price level induces households with negative wealth to

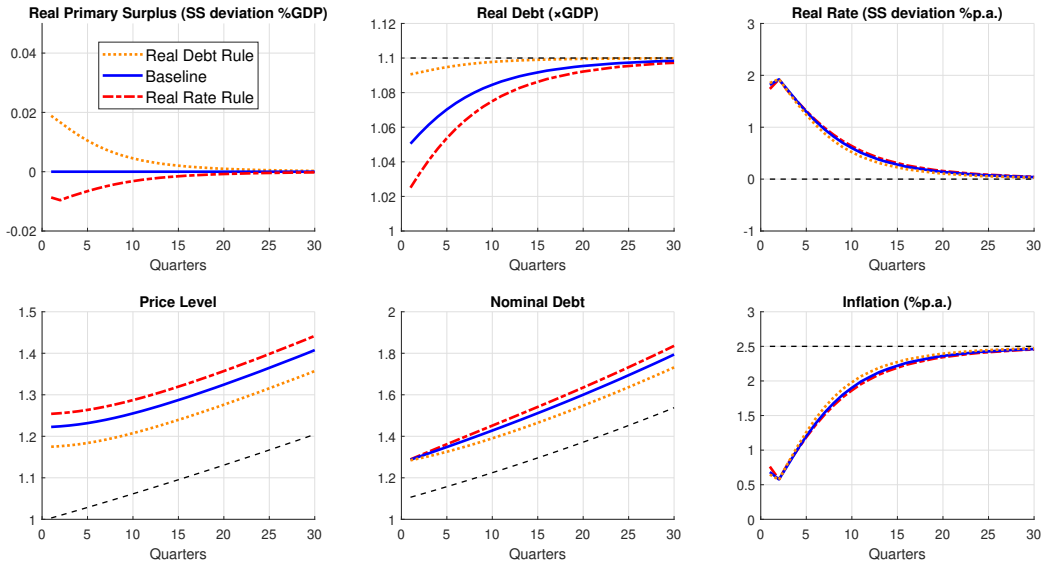
increase their consumption, because it lowers the real value of their debt. For households with positive wealth, the higher price level reduces their consumption because the real value of their nominal savings is curtailed. The higher real interest rate weakens consumption for all households because of an intertemporal motive, except for households on the borrowing constraint. The dashed black line delineates the winners and losers of this experiment in terms of 2019 US dollars. Households with assets lower than \$51,400, which account for 55% of the population in our calibrated economy, gain from the helicopter drop.

Targeted vs Untargeted Fiscal Helicopter Drop. Figure 7 also shows that initial increase in the price level is even larger when the the helicopter drop is targeted towards poorer households. Compared to the untargeted case, the real interest rate rises by 1 additional percentage point on impact and, as a result, the price level jumps by an additional 4 percentage points (to 25%). In both the untargeted and targeted cases, the fiscal helicopter drop has a permanent effect on the price level and nominal government debt, but the inflationary effects are temporary. The saddle-path dynamics imply that both the real interest rate and the inflation rate return to their initial levels. In these experiments, the different price level responses between the heterogeneous agent and representative agent economies are mostly in terms of timing. The higher initial rise in prices in the heterogeneous agent economy is followed by lower inflation, and the long-run cumulative increase in the price level is the same in the two economies.

Fiscal Helicopter Drop Under Different Surplus Reaction Rules. To justify focusing attention on the saddle-path equilibrium we are implicitly appealing to long-run inflation anchoring. As discussed in Section 4.3, surplus reaction rules are an alternative route to uniqueness. Figure 10 shows that the price level, real rate and inflation dynamics from the fiscal helicopter drop are not sensitive to using either of the two classes of surplus reaction rules in equations (33) and (34) that guarantee a unique equilibrium.

However, the two rules differ in the direction that primary deficits respond to the fiscal helicopter drop. Under the real debt reaction rule (33), the downward revaluation of real debt from the initial burst of inflation leads the fiscal authority to cut deficits following the helicopter drop. Under the real rate reaction rule, the

Figure 10



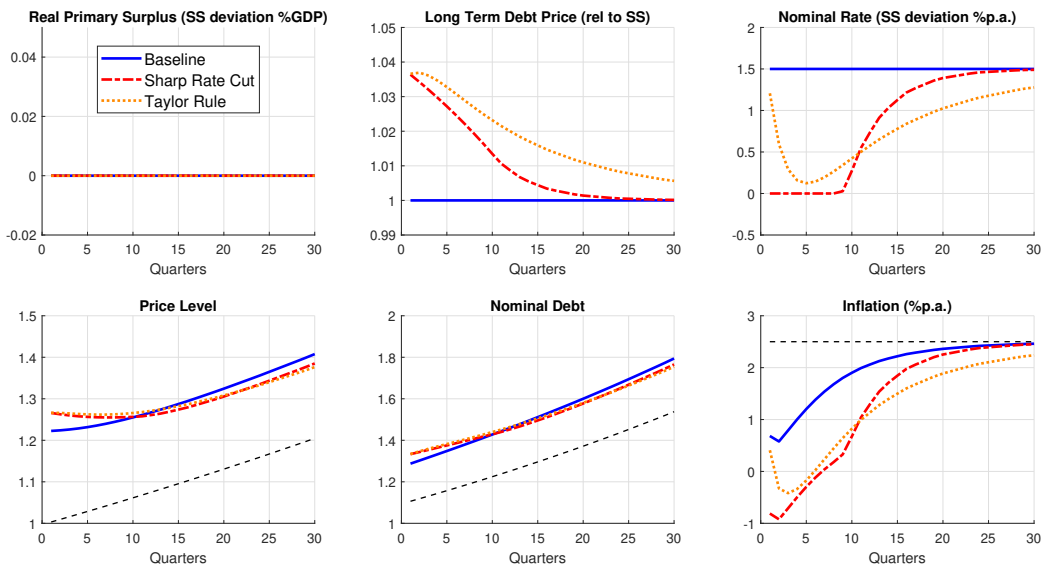
Note: Impulse responses to targeted fiscal helicopter drop under alternative fiscal rules. The dotted orange line corresponds to the “real debt rule” of equation equations (33) and the dashed red line corresponds to the “real rate rule” in equation (34) with parameter values of $\phi_b = -0.5$ and $\phi_r = -2$, respectively. The dashed black line plots the initial steady state.

higher real interest rate leads to a temporary increase in primary deficits.³⁷

Fiscal Helicopter Drop Under Different Monetary Responses. Throughout our previous simulations we have assumed that the central bank holds the nominal rate constant at 1.5% in response to the helicopter drop. Figure 11 reports results from two alternative experiments in which nominal rates are lowered at the same time as the fiscal expansion, like was done by central banks around the world in 2020. The dotted orange line labelled “Taylor rule” shows the effects of following a lagged Taylor rule as in equation (22), with a feedback parameter $\theta_m = 1$ and a coefficient on inflation $\phi_m = 0.5$. The dashed red line labelled “sharp rate cut” shows the implication of an even more powerful monetary accommodation of the fiscal expansion, corresponding to an immediate cut in the short-term interest rate all the way to zero, followed by a gradual normalization after 9 quarters. For comparison, the blue line labelled “baseline” reproduces the dynamics holding the nominal rate constant.

³⁷Cochrane (2023) argues that following an expansion in nominal debt, a reduction in primary deficits is more in line with the historical record for the U.S. However, Jacobson et al. (2023) discuss an important historical example in which new debt was issued with the explicit intention of generating inflation by committing to not raise future surpluses to repay the debt.

Figure 11

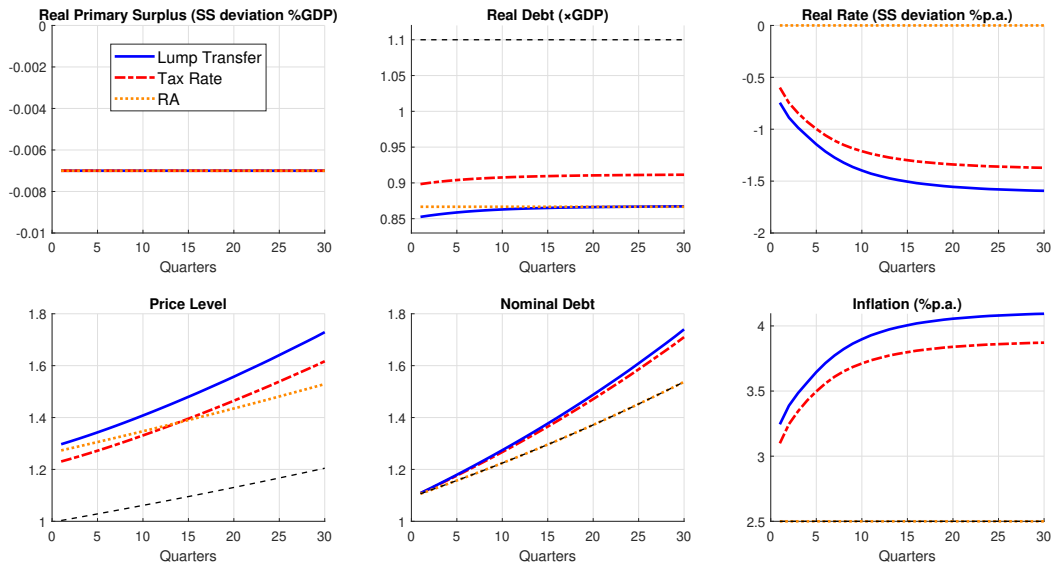


Note: Impulse response to targeted fiscal helicopter drop under different monetary policy responses. The dotted orange line corresponds to the Taylor rule in equation (22) with $\theta_m = 1$ and $\phi_m = 0.5$. The dashed red line is a temporary cut of nominal rates all the way to the zero lower bound. The dashed black line plots the initial steady state.

Monetary policy is a crucial driver of nominal aggregates. The behavior of long-term government bond prices is central to these dynamics. As explained in Sims (2011) and Cochrane (2018), a lower short-term nominal rate leads, through the yield curve, to a higher price of long-term government bonds. Thus, the overall price level must rise by a larger amount to achieve the same-size drop in the real value of outstanding government debt. Figure 11 shows that looser monetary policy causes an additional 4 to 6 percentage point increase in the price level upon impact, relative to the baseline with a nominal rate peg. The strength of this force is determined by the average duration of debt: the longer the duration, the bigger the initial jump in the price level. Different jumps in the price level, in turn, lead to different dynamics for real government debt and real interest rates through their effect on the real wealth distribution. However, we have found the effect on real variables to be quantitatively very similar across the three monetary specifications, provided that it is higher-wealth households that hold assets of longer duration.³⁸

³⁸If higher wealth households have longer duration portfolios, an unanticipated increase in monetary policy leads to larger capital losses for high-wealth households. However for moderate movements in the nominal rate, the relatively low MPCs of these households lead to small movements in the real rate. The assumption that high-wealth households hold relatively higher duration assets is

Figure 12



Note: Impulse response to a permanent expansion in primary deficits. The dotted orange line shows the effects of a reduction in surplus in the Representative Agent model. The blue line labelled “Lump Sum” illustrates the dynamics following an expansion of lump sum transfers. The dashed red line labelled “Tax Rate” plots dynamics following a tax cut. The orange line plots dynamics in the representative agent (RA) model. The dashed black line plots the initial steady state.

5.5 Permanent Deficit Expansion

Figure 12 displays impulse responses to a permanent deficit expansion from 3.3% to 4% of GDP. We consider two alternative policies for achieving a higher level of deficits. The solid blue line labeled “Lump-Sum” keeps the tax rate the same and raises the lump-sum transfer. The dashed red line labeled “Tax Rate” reduces the proportional tax rate, while keeping lump-sum transfers at their initial level.

As was shown in Figure 3, a permanent increase in deficits shifts the steady-state government nullcline downwards and to the right. Starting from the high real rate, low inflation steady-state, the long-run impact of the deficit expansion is to permanently lower both the real rate and the real value of government debt. These effects can be seen in the top row of Figure 12. The reduction in the value of real debt is achieved through a jump in the price level. In addition, because monetary policy does not respond, the lower real-rate translates into a permanently higher inflation rate. To prevent the permanent increase in deficits from leading to permanently

consistent with empirical evidence. See Doepke and Schneider (2006); Greenwald et al. (2021).

higher inflation, the central bank would need to track the fall in the real rate by decreasing its nominal rate target.

Hence in the heterogeneous agent economy with deficits and negative real rates, a secular increase in primary deficits can account for a secular decline in real rates, i.e. secular stagnation. The fact that permanently higher deficits result in a permanently lower real rate and higher inflation is a distinguishing feature of the heterogeneous agent economy relative to the representative agent economy, in which a permanent increase in deficits has no impact on real rates or inflation.

These effects are all more pronounced when deficits are increased by raising lump-sum transfers than by lowering the proportional tax rate. The reason is that raising lump-sum transfers lowers the amount of uninsured idiosyncratic risk, thereby weakening the overall precautionary motive in the economy, while lowering proportional taxes raises the overall precautionary motive. Graphically, these differences manifest as different shifts in the household asset demand curve $\mathbf{a}(r)$.

5.6 Additional Quantitative Results

Inflationary Effects of Redistributive Wealth Taxes. In order to emphasize the inflationary effects that arise from redistribution, Not For Publication Appendix [K](#) considers purely redistributive shocks: one-time wealth taxes levied on the top 10% of the wealth distribution, the proceeds of which are redistributed lump-sum to the bottom 60%. Although these shocks do not entail any new issuance of government debt or any change in primary deficits, they do cause a prolonged period of inflation.

Endogenous Output. Not For Publication Appendix [L](#) studies a permanent change in primary deficits in an economy where households make a labor-leisure choice with endogenous output. This extension serves to demonstrate that none of the qualitative forces relating heterogeneity and precautionary savings to prices and inflation that we have emphasized depend on an endowment economy per se.

6 Conclusions

We extend the fiscal theory of the price level to a heterogeneous-agent incomplete-market economy with flexible prices. In contrast to its representative agent counterpart, this model can be used to study an environment in which the government

runs persistent deficits and the real rate is below the aggregate growth rate of the economy. This configuration is a more accurate representation of the current state of affairs in many developed economies.

After showing that this model generically has two steady-states, we proposed a number of ways to obtain uniqueness for price level and inflation dynamics. Armed with uniqueness, we performed experiments that illustrate the forces at work in our model. The feature of our economy that accounts for different dynamics relative to its representative agent counterpart is the two-way feedback between price-level dynamics on the one hand, and redistribution and precautionary saving on the other. Redistribution and precautionary saving are also key determinants of the maximum deficit the economy can permanently sustain.

In on-going work we are extending this framework in two directions. The first is to include nominal rigidities, which gives rise to smoother price level dynamics. It also offers us the possibility to quantitatively confront the FTPL with the joint dynamics of inflation and output observed in the data, along the lines of what [Bianchi et al. \(2023\)](#) did in a representative agent model. The second is to extend our model to a two-asset economy with both low return nominal government bonds, and higher return real productive assets. Incorporating a two-asset household sector as in [Kaplan et al. \(2018\)](#) opens the door to a quantitative framework with a richer characterization of the possible assets through which households can save.

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Online Appendix

A Representative Agent Economy

A.1 RA: Environment

Notation closely follows that of the main text. There exists a representative household that chooses real consumption flows \tilde{c}_{jt} to maximize

$$\int e^{-\rho t} \frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma} dt \quad (\text{A.1})$$

Initial nominal assets A_0 are given. The household faces a flow budget constraint

$$dA_t = [i_t A_t + (1 - \tau_t) P_t y_t - P_t \tilde{c}_t] dt \quad (\text{A.2})$$

subject to the borrowing constraint $A_t \geq 0$, where τ_t is a path of taxes set by the government. We may express the budget constraint in real de-trended terms as

$$da_t = [r_t a_t + (1 - \tau_t) - c_t] dt \quad (\text{A.3})$$

where the real rate is defined as $r_t := i_t - \pi_t - g$. Government debt dynamics follow

$$db_t = [r_t b_t - \tau_t] dt \quad (\text{A.4})$$

We also impose the commonly maintained assumption in the fiscal theory of the price level that the government can borrow, but not lend: $b_t \geq 0$.

Household Optimality. It is easy to show that the solution to the representative household problem yields the Euler equation

$$\rho - r_t = -\gamma \frac{1}{c_t(a_t)} \frac{dc_t(a_t)}{dt} \quad (\text{A.5})$$

together with the household's transversality condition

$$\lim_{t \rightarrow \infty} e^{-\rho t} c^{-\gamma} a_t \leq 0 \quad (\text{A.6})$$

Monetary Policy. We allow for arbitrary monetary policy rules i_t , but assume that they lead to well-defined paths for inflation given real rates r_t (see Section 2.3).

A.2 RA: Equilibrium Definition

We now define a real equilibrium under the assumption that the price level P_t is differentiable for all $t > 0$.

Definition 2. *A real equilibrium is a collection of variables $\{c_t, a_t, b_t, r_t\}_{t \geq 0}$ such that:*

1. For all $t > 0$, c_t satisfies the Euler equation (A.5) and transversality condition (A.6).
2. For all $t > 0$, a_t evolves according to the budget constraint (A.3).
3. For all $t > 0$, b_t evolves according to the government budget constraint (A.4).
4. For all $t \geq 0$, markets clear: $a_t = b_t$.

Note that by Walras' Law, $c_t = 1$ for all $t \geq 0$ so that the goods market clears.

A.3 RA: Uniqueness With Constant Surpluses

Next, we show that a real unique equilibrium exists whenever $\tau = \tau^* > 0$, so that the government is running constant surpluses. First, note that the Euler equation (A.5) along with market clearing for output $c_t = 1$ implies that $r_t = \rho$ for all $t \geq 0$. Integrating the government budget constraint forwards then yields (A.4):

$$b_0 = \lim_{T \rightarrow \infty} \left[\int_0^T e^{-\rho t} \tau^* dt + e^{-\rho T} b_T \right] \quad (\text{A.7})$$

By transversality (A.6) and market clearing, the latter term must be non-positive. Moreover, it cannot be negative as this would violate the non-negativity constraint on household assets and/or the assumption that the government cannot be a lender. Hence, it must be zero. But this then implies that

$$b_0 = \lim_{T \rightarrow \infty} \left[\int_0^T e^{-\rho t} \tau^* dt \right] = \frac{\tau^*}{\rho} \quad (\text{A.8})$$

so b_0 is well-defined and strictly positive for any level of initial nominal assets B_0 . The dynamics for real debt $\{b_t\}_{t > 0}$ are then pinned down by the government budget constraint (A.4). This proves the existence of a unique real equilibrium.

Given an initial level of nominal debt B_0 , uniqueness of the real equilibrium implies uniqueness of the initial price level P_0 . Subsequent inflation is uniquely pinned down by $r_t = \rho$, and a monetary policy rule which sets the path for the nominal rate i_t .

The Case of Deficits. The analysis above requires that the present discounted value in (A.8) be finite and positive. Hence, running persistent deficits cannot be an admissible equilibrium under the assumption that (i) households face borrowing constraints or (ii) that aggregate government debt must be non-negative.

B Representative Agent with Bonds-In-Utility

B.1 RA-BIU: Environment

Our notation follows closely that of the main text. Time is continuous and indexed by t . The economy is populated by a representative agent that derives utility from consumption streams c_t and real asset holdings a_t according to:

$$\int e^{-\rho t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \zeta \ln(a_t + \underline{a}) \right) dt \quad (\text{B.1})$$

where $\rho > 0$ denotes the household's discount rate, and ζ , \underline{a} are positive constants. Our assumption that real assets enter utility in a logarithmic fashion is inessential to the main results. However, logarithmic utility will allow us to characterize the steady-states of the economy in closed-form.

As derived in Section A.1 at all points in time in which the price level is differentiable, the household budget constraint can be written in real terms as follows:

$$da_t = [r_t a_t + (1 - \tau_t)y_t - c_t] dt \quad (\text{B.2})$$

where $r_t = i_t - \pi_t$ denotes the real interest rate on bonds. The government budget constraint can similarly be written in real terms as:

$$db_t = [r_t b_t - \tau_t y_t] dt \quad (\text{B.3})$$

We also impose the commonly maintained assumption in the fiscal theory of the price level that the government can borrow, but not lend: $b_t \geq 0$.³⁹ To simplify the exposition, in this section only, we assume zero growth. Positive growth is straightforward to incorporate by letting ζ grow over time at the appropriate rate.

³⁹As explained in the context of the RA model of Section A, this can also be rationalized through a borrowing constraint on the household side.

Household Optimality. The representative household takes the future sequence of real rates r_t and output y_t as given, and chooses consumption and real asset holdings optimally subject to its budget constraint (B.2). This implies the following Euler Equation

$$\frac{1}{c_t} \frac{dc_t}{dt} = \frac{1}{\gamma} \left(r_t - \rho + \frac{\zeta c_t^\gamma}{a + \underline{a}} \right) \quad (\text{B.4})$$

The household must also satisfy the following transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\rho t} c_t^{-\gamma} a_t \leq 0 \quad (\text{B.5})$$

B.2 RA-BIU: Equilibrium Definition

The definition of equilibrium for this model is exactly as in Section A.2, with the exception that the Euler equation is given by (B.4).

Price Level Determination. As in the main text, each real equilibrium defines a unique price level determined by:

$$P_0 = \frac{B_0}{b_0} \quad (\text{B.6})$$

The path of inflation is then determined residually through the Fisher identity $\pi_t = i_t - r_t$. We assume for simplicity a monetary policy peg, $i_t = i^*$, but note that all our results on equilibrium uniqueness extend to the more general monetary rules outlined in Section 2.3.

B.3 RA-BIU: Uniqueness with Constant Surpluses

We now show that a unique real equilibrium exists under a constant, strictly positive surplus rule $\tau_t = \tau^*$, where $\tau^* > 0$.

Proposition 1. *A unique real equilibrium exists. Moreover, $r_t = r^*$ and $b_t = b^*$ for all $t \geq 0$, where r^* and b^* are strictly positive constants that are given by:*

$$r^* = \begin{cases} \frac{-(\tau^* + \zeta - \rho \underline{a}) + \sqrt{(\tau^* + \zeta - \rho \underline{a})^2 + 4\rho \underline{a} \tau^*}}{2\underline{a}} & \text{if } \underline{a} > 0 \\ \frac{\rho \tau^*}{\tau^* + \zeta} & \text{if } \underline{a} = 0 \end{cases} \quad (\text{B.7})$$

and

$$b^* = \frac{\tau^*}{r^*} \quad (\text{B.8})$$

Proof. Our proof proceeds in several steps.

Step 1: Monotonicity of real assets. We first show that a_t is increasing if $a_t > a^*$ and decreasing if $a_t < a^*$, where $a^* > 0$ is the unique steady-state value of real debt. We then show that this implies that $a_t < a^*$ at any t violates the non-negativity constraint on debt. Finally, we show that $a_t > a^*$ at any t is inconsistent with household optimality.

First, note that equation (B.4) together with $c_t = 1$ at all t implies that the real rate is given by the following equation for all t

$$r_t = \rho - \zeta \frac{1}{a_t + \underline{a}} \quad (\text{B.9})$$

Imposing market clearing and using the government budget constraint (B.3), we can derive an expression for the dynamics of real debt

$$\dot{a}_t = \left(\rho - \zeta \frac{1}{a_t + \underline{a}} \right) a_t - \tau^* \quad (\text{B.10})$$

where $\dot{a}_t \equiv \frac{da_t}{dt}$. The steady-states of this differential equation are given by

$$\frac{\tau^*}{a^*} = \rho - \zeta \frac{1}{a^* + \underline{a}} \quad (\text{B.11})$$

Note that the left-hand side of the above equation is decreasing in a^* whenever $\tau^* > 0$ (and asymptotes to zero as $a^* \rightarrow \infty$ and infinity as $a^* \rightarrow 0$), while the right-hand side is increasing in a^* (and asymptotes to $\rho > 0$ as $a^* \rightarrow \infty$). Moreover, both terms are continuous for $a^* > 0$. Hence, a unique steady-state with a strictly positive real rate exists. Denote this real rate by $r^* > 0$.

Further, \dot{a}_t is strictly positive whenever $a_t \in (a^*, \infty)$ and strictly negative whenever $a_t \in (0, a^*)$. Suppose otherwise. We have that:

$$\frac{d\dot{a}_t}{da} \Big|_{a_t=a^*} = r^* + \zeta a^* (a^* + \underline{a})^{-2} > 0 \quad (\text{B.12})$$

Moreover, \dot{a}_t is continuously differentiable on $a_t > 0$. Hence, $\dot{a}_t(a_t) < 0$ for some $a_t \in (a^*, \infty)$ would imply that there exists an $a^{**} \in (a^*, \infty)$ such that $\dot{a}_t(a^{**}) = 0$, thereby violating steady-state uniqueness.

Step 2: Ruling Out Downwards Explosions. Next, we rule out all equilibria in which $a_{t'} < a^*$ for some $t' \geq 0$. By Step 1, $a_{t'} < a^*$ implies that $a_{t'} < a_t$ for all $t' \geq t$. Moreover, there are no limit points such that $\lim_{t \rightarrow \infty} a_t = a^{**}$ for any $a^{**} > 0$. Hence, any path in which $a_{t'} < a^*$ implies that the constraint $a_t > 0$ must be violated in finite time.

Step 3: Ruling Out Upwards Explosions. We now rule out equilibria in which $a_{t'} > a^*$ for some $t' \geq 0$. We may integrate the government budget constraint (B.3) forwards to obtain

$$a_0 = \lim_{T \rightarrow \infty} \left[\int_0^T \exp \left(\int_0^s -r_u du \right) \tau^* ds + \exp \left(\int_0^T -r_u du \right) a_T \right] \quad (\text{B.13})$$

Note that $r_t > 0$ whenever $a_t > a^*$, so the first term in the limit is well-defined. Moreover, (B.3) implies that assets will be growing at rate r_t whenever $a_t \geq a^*$. Hence, the second-term is non-zero if and only if $a_{t'} \geq a^*$ for some $t' \geq 0$.

We now show that household optimality implies that this second term must necessarily be finite. Substituting for the real rate, we obtain:

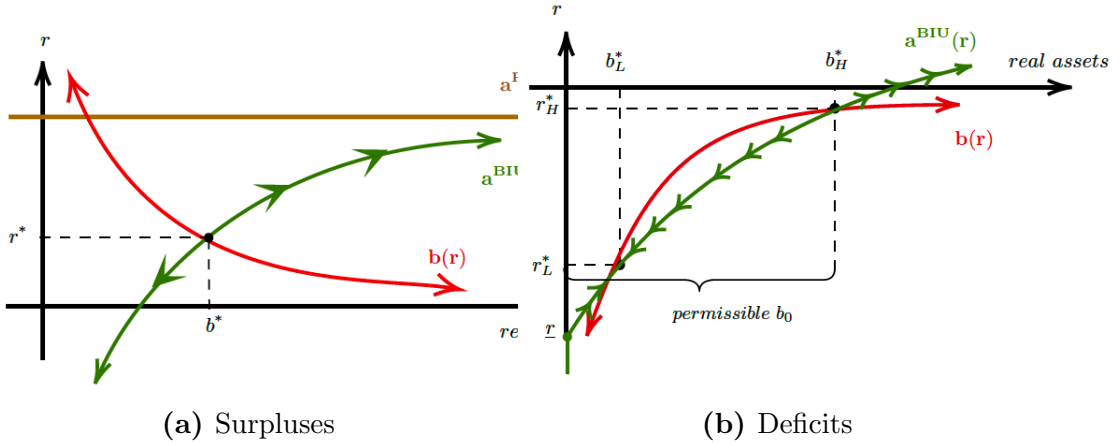
$$\lim_{T \rightarrow \infty} \left[\exp(-\rho T) a_T \times \exp \left(\int_0^T \zeta \frac{1}{a_u + \underline{a}} du \right) \right] \quad (\text{B.14})$$

The first-term in this expression is zero by the transversality condition (B.5). The second term is bounded as assets are growing at an exponential rate. Hence, we must have $a_t = a^*$ and $r_t = r^*$ for all $t \geq 0$. Equation (B.13) then implies that the second term is zero and thus a_0 must be given by

$$a_0 = \int_0^\infty \exp \left(\int_0^s -r^* du \right) \tau^* ds = \frac{\tau^*}{r^*} = a^* \quad (\text{B.15})$$

Substituting for a^* in (B.9) yields a quadratic equation with a unique, strictly positive root given by (B.7). This completes the proof. □

The intuition for this result closely mirrors that of the representative agent economy. The system of equations (B.3) and (B.4) are globally unstable. Paths in which $b_0 < b^*$ therefore lead to downward explosions, which violate the non-negativity condition on debt. Paths in which $b_0 > b^*$ lead to an excessive accumulation of assets,



Note: Dynamics for real assets in the RA-BIU economy when $\tau^* > 0$ (left panel) and $\tau^* < 0$ (right panel), as given by (B.3) and (B.4)

thereby violating household optimality. These dynamics are graphically depicted in Figure 13a. Note that r^* is strictly increasing in τ^* , with $r^* \rightarrow \rho$ and $a^* \rightarrow \infty$ as $\tau^* \rightarrow \infty$. In this sense, the steady-state asset demand in the RA-BIU economy (B.18) exhibits many similar features to the heterogeneous agent economy considered in the main text.

B.4 RA-BIU: Dynamics with Constant Deficits

Next, we consider dynamics under constant deficits $\tau^* < 0$. We show that the price level is generally no longer determinate for a given value of initial nominal debt. Intuitively, the steady-states of the government accumulation equation (B.3) form an *upward* sloping locus in $r - b$ space, as depicted graphically in Figure 13b. This can give rise to steady-state multiplicity, eliminating the explosive dynamics that are required in order to obtain uniqueness. The following proposition formally characterizes the nature of this steady-state multiplicity.

Proposition 2. *Suppose $\rho \underline{a} < \zeta$. Then:*

1. *If $\underline{a} = 0$, a unique steady-state exists if $\tau^* \in (-\zeta, 0)$, and no steady-state exists if $\tau^* \in (-\infty, -\zeta]$.*
2. *If $\underline{a} > 0$, there exists a $\underline{\tau} \in (\rho \underline{a} - \zeta, 0)$ such that two distinct steady-states exist if $\tau^* \in (\underline{\tau}, 0)$, no steady-state exists if $\tau^* \in (-\infty, \underline{\tau})$, and a unique steady-state exists if $\tau^* = \underline{\tau}$.*

Proof. We may substitute for steady-state assets in (B.9) to obtain

$$r^* = \rho - \frac{\zeta}{\frac{\tau^*}{r^*} + \underline{a}} \quad (\text{B.16})$$

We may solve the above equation to express the steady-states of the system as:

$$r^* = \begin{cases} \frac{-(\tau^* + \zeta - \rho \underline{a}) \pm \sqrt{(\tau^* + \zeta - \rho \underline{a})^2 + 4\rho \underline{a} \tau^*}}{2\underline{a}} & \text{if } \underline{a} > 0 \\ \frac{\rho \tau^*}{\tau^* + \zeta} & \text{if } \underline{a} = 0 \end{cases} \quad (\text{B.17})$$

and

$$a^* = \frac{\tau^*}{r^*} \quad (\text{B.18})$$

where we additionally require $r^* < 0$ so that the non-negativity constraint on assets is not violated. It is straightforward to see that this condition is satisfied if and only if $\tau^* > -\zeta$ when $\underline{a} = 0$. This proves the first part of the proposition.

To prove the second part of the proposition, note a necessary and sufficient condition for $r^* < 0$ in the constant deficit economy is $\tau^* \in (\rho \underline{a} - \zeta, 0)$ and that $(\tau^* + \zeta - \rho \underline{a})^2 + 4\rho \underline{a} \tau^* > 0$. This is negative at $\tau^* = \rho \underline{a} - \zeta$, positive at $\tau^* = 0$, and strictly increasing on $(\rho \underline{a} - \zeta, 0)$. Hence, there exists a unique root of this expression within this interval given by $\underline{\tau} \in (\rho \underline{a} - \zeta, 0)$. It follows that there are two distinct steady-states whenever $\underline{\tau} < \tau^* < 0$, no steady-states whenever $\tau^* < \underline{\tau}$ and a unique steady state whenever $\tau^* = \underline{\tau}$. \square

The condition $\rho \underline{a} < \zeta$ ensures that there exists a negative interest rate such that households are willing to hold strictly positive amounts of real assets (no steady-state with deficits exists trivially if this condition is not satisfied). Note also that, depending on the value of \underline{a} , zero, one, or two equilibria can exist. Further, at least one equilibrium exists as long as the level of deficits is not too large. We next show how steady-state multiplicity is tied to price level determinacy. In particular, a unique equilibrium exists if and only if a unique steady-state exists.

Proposition 3. *The following statements are true.*

1. *If no steady-state exists, then no real equilibria exist.*
2. *If a unique steady-state exists, then there exists a unique real equilibrium with constant real rates $r_t = r_H^* = r_L^*$ and real assets $b_t = b_H^* = b_L^*$.*

3. *If two distinct steady-states exist, then there exists a continuum of real equilibria indexed by $b_0 \in (0, b_H^*]$.*

Proof. Suppose no steady-states exist. Equation (B.3) then implies that real assets will tend to infinity or minus infinity for any given b_0 . The former case is ruled out, as it violates the transversality condition by Proposition 1. The latter case is ruled out as it implies that assets will violate their non-negativity constraint in finite time. Hence, no equilibria exist.

Next, suppose that a unique steady-state exist. Define the function

$$r(a_t) = \rho - \frac{\zeta}{a_t + \underline{a}} \quad (\text{B.19})$$

From (B.3), steady-states are given by the roots to

$$g(a) = r(a) - \frac{\tau^*}{a} \quad (\text{B.20})$$

There exists a unique a^* such that $g(a^*) = 0$ by assumption. Moreover, $g(a) \rightarrow \rho > 0$ as $a \rightarrow \infty$, so we must have $g'(a^*) > 0$ by the intermediate value theorem. Using the government accumulation equation, the dynamics of real debt around a^* are given by

$$\begin{aligned} d\frac{\tilde{a}_t}{dt} &= [r'(a^*)a^* + r(a^*)]\tilde{a}_t \\ &= \left[r'(a^*) + \frac{\tau^*}{(a^*)^2}\right]\frac{\tilde{a}_t}{a^*} = g'(a^*)\frac{\tilde{a}_t}{a^*} > 0 \end{aligned}$$

where $\tilde{a}_t = a_t - a^*$, to first-order. Because a^* is unique by assumption, real assets explode upwards exponentially at a rate r_t when $a_0 > a^*$ (violating (B.5)) and downwards when $a_0 < a^*$ (violating the non-negativity of assets in finite time). Hence, a unique equilibrium exists.

Suppose now that two equilibria exist $a_H^* > a_L^*$. The top equilibrium is locally unstable by the argument presented above. The bottom equilibrium is locally stable, since $g(a) \rightarrow \infty$ as $a \rightarrow \infty$. Hence, $g'(a_L^*) < 0$. This implies that all equilibria with $b_0 \in (0, b_H^*)$ converge to b_L^* , while all equilibria with $b_0 > b_H^*$ feature explosive dynamics that violate (B.5). Thus, there exist a continuum of equilibria indexed by $b_0 \in (0, b_H^*]$. \square

One can show that the presence of two steady-states imply a non-singular basin

of attraction for the economy. Hence, a continuum of real equilibria, indexed by their initial condition b_0 , are possible. Note that the final condition places a lower bound on the price level for any given level of initial nominal assets, given by $P_0 = \frac{B_0}{b_H^*}$.

C Additional Derivations

C.1 Derivation of Optimal Consumption Dynamics

This section derives expressions for the consumption dynamics of unconstrained constrained households.

Unconstrained Households. We show that the expected consumption dynamics for unconstrained households are given by

$$\frac{\mathbb{E}_t [dc_{jt}]}{c_{jt}} = \frac{1}{\gamma} (r_t - \rho) dt + \frac{1}{\gamma} \sum_{z'} \lambda_{z_j t z'} \left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}} \right)^{-\gamma} dt + \sum_{z'} \lambda_{z_j t z'} \left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}} \right) dt. \quad (\text{C.1})$$

Here we use the short-hand notation $c_{jt} := c(a_{jt}, z_{jt}, \Omega_t)$ to denote the consumption of household j at time t . Recall the HJB Equation:

$$\rho V_t(a, z) = \max_c u(c) + s_t(a, z) \partial_a V(a, z) + \sum_{z' \neq z} \lambda_{z, z'} [V_t(a, z') - V_t(a, z)] + \partial_t V_t(a, z) \quad (\text{C.2})$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $s_t(a, z)$ is the savings function (11). The FOC is:

$$u'(c) = \partial_a V_t(a, z) \quad (\text{C.3})$$

Differentiating the above with respect to a yields

$$u''(c_t(a, z)) \partial_a c_t(a, z) = \partial_{aa}^2 V_t(a, z) \quad (\text{C.4})$$

Differentiating with respect to t yields

$$u'''(c_t(a, z)) \partial_t c_t(a, z) = \partial_{at}^2 V_t(a, z) \quad (\text{C.5})$$

The envelope condition for (C.2) is:

$$\rho \partial_a V_t(a, z) = \partial_{aa}^2 V_t(a, z) s_t(a, z) + r_t \partial_a V_t(a, z) + \sum_{z' \neq z} [\partial_a V_t(a, z') - \partial_a V_t(a, z)] + \partial_{at}^2 V_t(a, z) \quad (\text{C.6})$$

Using (C.4) and (C.5) into the equation above yields:

$$\begin{aligned} (\rho - r_t) u'(c_t(a, z)) &= \sum_{z' \neq z} \lambda_{zz'} [u'(c_t(a, z')) - u'(c_t(a, z))] \\ &+ u''(c_t(a, z)) [\partial_t c_t(a, z) + s_t(a, z) \partial_a c_t(a, z)] \end{aligned} \quad (\text{C.7})$$

(C.7) holds at any point on the interior of the state space $a > 0$ (i.e. for all unconstrained households). Using Ito's lemma for jump processes, we can write it as:

$$(\rho - r) u'(c_t(a_j, z_j)) = \frac{d\mathbb{E}[u'(c_t(a_j, z_j))]}{dt} \quad (\text{C.8})$$

where we suppress the dependence of a_{jt} and z_{jt} on t for notational simplicity. Furthermore, using Ito's lemma on $c_t(a_j, z_j)$ yields

$$\begin{aligned} dc_j &= \left[\partial_a c_t(a_j, z_j) s_t(a_j, z_j) + \partial_t c_t(a_j, z_j) + \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \right] dt \\ &+ [c_t(a_j, z') - c_t(a_j, z_j)] d\tilde{N}_j \end{aligned} \quad (\text{C.9})$$

where \tilde{N}_j is the compensated Poisson process for the stochastic process of income z' . Expected consumption therefore follows:

$$\mathbb{E}[dc_j] = \left[\partial_a c_t(a_j, z_j) s_t(a_j, z_j) + \partial_t c_t(a_j, z_j) + \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \right] dt \quad (\text{C.10})$$

We may combine this with (C.7) to obtain

$$\begin{aligned} (\rho - r_t) u'(c_t(a_j, z_j)) &= \sum_{z' \neq z_j} \lambda_{z_j z'} [u'(c_t(a_j, z')) - u'(c_t(a, z_j))] \\ &+ u''(c_t(a_j, z_j)) \frac{\mathbb{E}[dc_j]}{dt} - u''(c_t(a_j, z_j)) \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \end{aligned} \quad (\text{C.11})$$

This yields (C.1) after dividing by $u''(c_t(a_j, z_j))$ and specializing to $u'(c) = \frac{c^{1-\gamma}}{1-\gamma}$

Constrained Households. We show that the expected consumption dynamics for borrowing constrained households satisfy:

$$\frac{\mathbb{E}_t [dc_{jt}]}{c_{jt}} = \sum_{z'} \lambda_{z_{jt}, z'} \left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}} \right) dt. \quad (\text{C.12})$$

The consumption dynamics for constrained households are given by

$$dc_t(0, z_j) = \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(0, z') - c_t(0, z_j)] dt + [c_t(0, z') - c_t(0, z_j)] d\tilde{N}_j \quad (\text{C.13})$$

since households consume their income whenever constrained (until receiving a more favourable income draw). Taking expectations and dividing by $c_t(0, z_j)$ then yields (C.12) directly.

C.2 Existence of \underline{r}

This subsection shows that there exists a finite \underline{r} such that no household saves in a stationary equilibrium if $r \leq \underline{r}$. Suppose no such \underline{r} exists. Note that this implies that there must exist a non-zero mass of households that are unconstrained in any stationary equilibrium, for all $r < \rho$.

Proposition 2 in Achdou et al. (2022) shows that there exists a finite upper bound on the state space for assets in a stationary equilibrium. Moreover, $\underline{z} > 0$. Hence, marginal utility and consumption are bounded from above and are strictly greater than zero for all a_{jt} and z_{jt} . Equation (C.1) then implies that there must exist an \underline{r} such that

$$\frac{\mathbb{E}[dc_t(a_{jt}, z_{jt})]}{dt} < 0$$

for all households j that are unconstrained. But this would then imply that aggregate consumption must be decreasing, which would violate market clearing. Hence, there cannot exist a non-zero mass of households that are unconstrained in a stationary equilibrium with $r < \underline{r}$. But this implies the existence of such an \underline{r} , a contradiction.

C.3 Uniqueness with Constant Surpluses

In this section, we show that explosive paths for real assets are ruled out by the household transversality condition. Our proof strategy entails decomposing the expectation in (24) and aggregating across households to show that the rate of growth of aggregate assets is bounded below by the discount rate ρ .

Consider a strictly positive sequence of real rates $(r_t)_{t \geq 0}$. Recall that the transversality condition in the stochastic economy is:

$$\lim_{t \rightarrow \infty} [\mathbb{E}_t \exp(-\rho t) u'(c_t(a_{jt}, z_{jt})) a_{jt}] = 0 \quad (\text{C.14})$$

The household Euler equation gives us a differential equation for the evolution of *expected* marginal utility.

$$\frac{\mathbb{E}_0[du'(c_t(a_{jt}, z_{jt}))]}{u'(c_0(a_{j0}, z_{j0}))} = (\rho - r_t) dt \quad (\text{C.15})$$

We may solve this ordinary differential equation to obtain

$$\mathbb{E}_0[u'(c_t(a_{jt}, z_{jt}))] = u'(c_0(a_{j0}, z_{j0})) \exp\left(\rho t - \int_0^t r_s ds\right) \quad (\text{C.16})$$

We next decompose the expectation term in the household transversality condition:

$$\begin{aligned} \lim_{t \rightarrow \infty} [\mathbb{E}_0 \exp(-\rho t) u'(c_t(a_{jt}, z_{jt})) a_{jt}] &= \lim_{t \rightarrow \infty} [\exp(-\rho t) \mathbb{E}_0 [u'(c_t(a_{jt}, z_{jt}))] \mathbb{E}_0 [a_{jt}] \\ &\quad + \exp(-\rho t) \text{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt})] \end{aligned} \quad (\text{C.17})$$

where the covariance is conditional on the households' time-zero information set. We may substitute for the first term using (C.16) to obtain:

$$\lim_{t \rightarrow \infty} \left[\exp\left(-\int_0^t r_s ds\right) u'(c_0(a_{j0}, z_{j0})) \mathbb{E}_0 [a_{jt}] + \exp(-\rho t) \text{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt}) \right] = 0 \quad (\text{C.18})$$

We may also bound the covariance term via the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \exp(-\rho t) |\text{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt})| &\leq \exp(-\rho t) \sqrt{\text{Var}_0(u'(c_t(a_{jt}, z_{jt})))} \sqrt{\text{Var}_0(a_{jt})} \\ &\leq \exp(-\rho t) \frac{y_{min}^{-\gamma}}{2} \sqrt{\mathbb{E}_0[(a_{jt})^2]} \end{aligned}$$

where last the inequality has made use of the fact that $u'(c_{jt}) \leq y_{min}^{-\gamma}$ and the Popoviciu bound on variances (Bhatia and Davis, 2000). Finally, we provide a bound on the variance of individual asset holdings. If asset holdings are uniformly bounded, the bound is trivially zero. So we only need to concern ourselves with cases in which individual assets may diverge to infinity. In these cases, we can use standard results on the asymptotic behaviour of the consumption function to provide an upper bound on assets (Benhabib et al., 2015; Achdou et al., 2022). In particular, we have:

$$\lim_{a_{jt} \rightarrow \infty} \frac{\phi_t a_{jt}}{c_{jt}} = 1 \quad (\text{C.19})$$

where $\phi_t > 0$. We may then use the household budget constraint to show that assets grow at a rate $r_t - \phi_t$ asymptotically, which yields the bound

$$a_{jt} \leq \Xi \exp\left(\int_0^t (r_s - \phi_s) ds\right), \quad \text{a.s.} \quad (\text{C.20})$$

for some finite $\Xi > 0$. Using the Popoviciu inequality once again, we obtain

$$|\exp(-\rho t) \text{Cov}_0(u'(c_{jt}), a_{jt})| \leq \exp(-\rho t) \frac{y_{min}^{-\gamma}}{4} \Xi \exp\left(\int_0^t (r_s - \phi_s) ds\right) \quad (\text{C.21})$$

Under the assumption that there exists some $t' > 0$ such that $r_t \leq \rho$ for $t \geq t'$, the right-hand side vanishes as we take $t \rightarrow \infty$. Section G provides sufficient condition for $r_t < \rho$ for all $t \geq 0$.

We now show that (C.18) precludes explosive paths for real aggregate debt. In particular, we show that

$$\lim_{t \rightarrow \infty} \left[\exp\left(\int_0^t -r_s ds\right) a_t \right] = 0 \quad (\text{C.22})$$

where a_t is the amount of aggregate asset holdings in the economy at time t . To this

end, we integrate (C.18) over households to obtain:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left[\int_{a,z} \mathbb{E}_0 \exp(-\rho t) u'(c_0(a, z)) a dG_t(a, y) \right] \\ & \leq \lim_{t \rightarrow \infty} m \left[\exp \left(- \int_0^t r_s ds \right) \underbrace{\int_{a,y} \mathbb{E}_{a_0=a} [a_t] dG_t(a, z)}_{\bar{a}_t} \right] = 0 \end{aligned}$$

where m is an upper bound on marginal utility at $t = 0$: $u'(c_0^i(a^i)) \leq m \quad \forall a, y \in \text{supp } G_0(a, y)$, a.e. and where $G_t(\cdot, \cdot)$ is the distribution over assets and income at time t . Note that term in the integral in the second inequality is equal to aggregate asset holdings by the exact law of large numbers (Duffie and Sun, 2012). This shows that no equilibria exist in which government debt explodes upwards. Downward explosion paths are ruled out by the non-negativity constraint on aggregate real debt.

C.4 Finite Difference Approximation

We begin by deriving the Kolmogorov Forward Equation (KFE) for wealth shares. Note that the dynamics for wealth shares $\omega_{jt} = \frac{a_{jt}}{a_t}$ is given by

$$\frac{d\omega_{jt}}{\omega_{jt} dt} = \frac{da_{jt}}{a_{jt} dt} - \frac{db}{b_t dt} \quad (\text{C.23})$$

Using Equations (6) and (18) yields

$$\frac{d\omega_{jt}}{dt} = \omega_{jt} \left(\frac{r_t a_{jt} + z_{jt} - \tau_t(z_{jt}) - c_{jt}}{a_{jt}} - \frac{r_t b_t - s_t}{b_t} \right) \quad (\text{C.24})$$

$$\frac{d\omega_{jt}}{dt} = \frac{z_{jt} - \tau_t(z_{jt}) - c_{jt} + \omega_{jt} s_t}{b_t} \quad (\text{C.25})$$

This implies that the KFE for wealth shares is given by:

$$\partial_t f(\omega, z) = \mathcal{A}_\omega^*[f, b](\omega, z) + \mathcal{A}_z^*[f](z) \quad (\text{C.26})$$

where

$$\mathcal{A}_\omega^*[f, b](\omega, z) = \partial_\omega \left[f(\omega, z) \frac{z - \tau_t(z) - c_t(\omega, z; f, b) + \omega s_t}{b} \right] \quad (\text{C.27})$$

and

$$\mathcal{A}_z^*[f](z) = -f(\omega, z) \sum_{z' \neq z} \lambda_{zz'} + \sum_{z' \neq z} \lambda_{z'z} f(\omega, z') \quad (\text{C.28})$$

where we have made the dependence of the consumption function on aggregate state variables explicit. Note further that these operators are adjoint to underlying operators \mathcal{A}_ω and \mathcal{A}_z .

We may discretize the distribution $f(\omega, z)$ into $N = N_\omega \times N_z$ discrete points, where N_ω is a discrete grid for ω of width Δ_ω . We denote the discretized distribution as \mathbf{f} and write the dynamics of the joint system as

$$\frac{d\mathbf{f}}{dt} = \mathbf{A}_\omega[\mathbf{f}_t, b_t]^T \mathbf{f}_t + \mathbf{A}_z^T \mathbf{f}_t \quad (\text{C.29})$$

$$\frac{db}{dt} = \mathbf{r}[\mathbf{f}_t, b_t] b_t - s^* \quad (\text{C.30})$$

The interest rate functional $\mathbf{r}[\mathbf{f}_t, b_t]$ corresponds to the interest rate functional in Equation (25) where we have substituted for the discretized endowment share distribution. The matrix $\mathbf{A}_\omega[\mathbf{f}_t, b_t]$ is a finite difference approximation to $\mathcal{A}[f, b]$ using the appropriate upwind scheme (Achdou et al., 2022). Hence, it is a tridiagonal matrix which consists of the following terms:

$$\left\{ 0, -\frac{z - \tau_t(z) - c_t(\omega, z; \mathbf{f}, b) + \omega s_t}{b\Delta_\omega}, \frac{z - \tau_t(z) - c_t(\omega, z; \mathbf{f}, b) + \omega s_t}{b\Delta_\omega} \right\} \quad (\text{C.31})$$

The matrix \mathbf{A}_z is the Markov transition matrix for z in the product space $\omega \times z$. Note that it is not indexed by z because the operator \mathbf{A}_z is linear. The rows of both $\mathbf{A}_\omega[\mathbf{f}_t, b_t]$ and \mathbf{A}_z sum to zero to ensure that \mathbf{f}_t preserves mass.

The linearized system can be exactly expressed as (31) if the effect of \mathbf{f} on the interest rate is small. A sufficient condition is that the real interest rate is invariant to changes in the endowment share distribution, which would occur if consumption functions were linear in wealth. However, because the interest rate functional uses a consumption-based aggregator, in practice it is only necessary for the consumption function to be linear amongst high-wealth households, who consume relatively more of the aggregate endowment.

C.5 Uniqueness with Zero Surpluses

The government accumulation equation with zero surpluses is

$$db_t = [\mathbf{r}(\Omega_t)b_t]dt \quad (\text{C.32})$$

This implies a steady-state interest rate of $r^* = 0$ whenever $\mathbf{a}(0) > 0$, with an associated steady-state level of real debt given by $b^* \equiv \mathbf{a}(0)$. The first-order dynamics of this system around the steady-state are given by:

$$db_t = [b^* \partial_b \mathbf{r}(\Omega^*)]dt \quad (\text{C.33})$$

The last term is strictly positive due to household behaviour. Hence, the steady-state is locally saddle-path stable. Since $B_0 > 0$ is given, there exists a unique, finite value of P_0 such that the equilibrium converges back to the steady-state. There are also a continuum of stationary real equilibria with $P = \infty$, in which $r < \underline{r}$ and aggregate real debt is zero. This proves local uniqueness of the equilibrium. Conditions for global uniqueness are outlined in Online Appendix C.3.

C.6 Steady-State Welfare Comparison

We show that steady-states with higher real interest rates are Pareto ranked for any initial condition of assets a_{jt} and income z_{jt} . In particular, consider a particular profile of income shocks $\{z_{jt}\}_{t \geq 0}$ that induces a (realized) consumption and savings streams $\{c_{jt}, a_{jt}\}$ under a constant real interest rate r_L^* . This consumption plan can also be implemented at a higher interest rate $r_H^* > r_L^*$ for the same sequence of income shocks, since the change in savings in any given period will be:

$$da_{jt} = [(r_H^* - r_L^*)a_{jt}]dt \quad (\text{C.34})$$

which is weakly positive for any given $a_{jt} > 0$ (recall that the surplus s^* , and hence taxes and transfers, are fixed and independent of the level of the real interest rate). Higher interest rates weakly expand the budget set of all households for any given a_{j0} and z_{j0} . This proves that a steady-state with r_H^* Pareto dominates r_L^* .⁴⁰

⁴⁰This proof strategy follows [Aguilar et al. \(2021\)](#), who construct robust Pareto-improving policies in the presence of capital accumulation.

C.7 Unique Steady State with Real Debt Reaction Rule

Our argument for uniqueness proceeds in three steps. First, we derive conditions for a unique steady-state. Second, we derive conditions for the steady-state to be saddle-path stable. This ensures local uniqueness. Finally, we consider whether explosive paths in debt can be ruled out globally. This ensures global uniqueness.

Steady-State Uniqueness. Suppose the government follows a fiscal rule of the form:

$$s_t = s^* + \phi_b(b_t - b^*) \quad (\text{C.35})$$

where $s^* = r^*b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$db_t = [r_t b_t - s_t]dt \quad (\text{C.36})$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{s^* - \phi_b b^*}{b} + \phi_b \quad (\text{C.37})$$

A sufficient condition for steady-state uniqueness is that this function is downwards sloping. This will ensure that it intersects the upwards sloping steady-state demand curve $\mathbf{a}(r)$ exactly once. The slope of this function is

$$\frac{dr}{db} = -\frac{s^* - \phi_b b^*}{b^2} \quad (\text{C.38})$$

$$= -\frac{r^* b^* - \phi_b b^*}{b^2} \quad (\text{C.39})$$

which is strictly negative whenever $r^* > \phi_b$. Hence, $\phi_b < r^* < 0$ is sufficient for steady-state uniqueness.

Local Uniqueness. We now examine conditions for this fiscal rule to give rise to local uniqueness. Under our maintained assumptions on the dynamical system that obtain (31), local uniqueness amounts to checking whether the eigenvalues of the government accumulation equation are strictly positive. The equilibrium dynamics are:

$$db_t = [(\mathbf{r}(\Omega_t) - \phi_b)b_t - (r^* - \phi_b)b^*]dt \quad (\text{C.40})$$

The first-order dynamics of this system around the steady-state are given by:

$$db_t = [\mathbf{r}(\Omega^*) - \phi_b + b^* \partial_b \mathbf{r}(\Omega^*)] dt \quad (\text{C.41})$$

The last term is positive because of household behavior. The sum of the first two terms are positive under the condition $r^* > \phi_b$. This proves local uniqueness.

Global Uniqueness. We now show that explosive dynamics are incompatible with equilibrium. Online Appendix C.3 shows that a sufficient condition for explosive dynamics to be inconsistent with equilibrium is for real debt to grow at a rate greater than r_t . But this follows from Equation (C.40) and $\phi_b < 0$.

C.8 Unique Steady State with Real Rate Reaction Rule

Our argument for uniqueness proceeds in three steps, as before.

Steady-State Uniqueness. Suppose the government follows a fiscal rule of the form:

$$s_t = s^* + \phi_r(r_t - r^*) \quad (\text{C.42})$$

where $s^* = r^* b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$db_t = [r_t b_t - s_t] dt \quad (\text{C.43})$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{(b^* - \phi_r)r^*}{b - \phi_r} \quad (\text{C.44})$$

Our goal is to obtain an upward sloping function for the null-cline that intersects the r -axis above $\mathbf{a}(r)$. This will ensure that it intersects the upwards sloping steady-state demand curve $\mathbf{a}(r)$ exactly once, as in Figure 5b. The slope of this function is

$$\frac{dr}{db} = -\frac{(b^* - \phi_r)r^*}{(b - \phi_r)^2} \quad (\text{C.45})$$

which is strictly positive whenever $b^* > \phi_r$. We also want the null-cline to intersect the r -axis at a negative real interest rate that is greater than \underline{r} (c.f. Figure 5b). This

occurs if

$$\phi_r < \frac{s^*}{r^* - \mathbf{a}^{-1}(0)} \quad (\text{C.46})$$

Local Uniqueness. We now examine conditions for this fiscal rule to yield local uniqueness. Under our maintained assumptions on the dynamical system that obtain (31), local uniqueness amounts to checking whether the eigenvalues of the government accumulation equation are strictly positive. The equilibrium dynamics are:

$$db_t = \mathbf{r}(\Omega_t)(b_t - \phi_r) - (r^* - \phi_r)b^*]dt \quad (\text{C.47})$$

The first-order dynamics of this system around the steady-state are given by:

$$db_t = [\mathbf{r}(\Omega^*) + (b^* - \phi_r)\partial_b\mathbf{r}(\Omega^*)]dt \quad (\text{C.48})$$

Note that at the top-right steady-state, we must have

$$\mathbf{r}'(\Omega^*) > -\frac{r^*}{b^*} \quad (\text{C.49})$$

which ensures that a sufficient condition for the right-hand side of (C.48) to be positive is $\phi_r < 0$. Hence, $\phi_r < 0$ is a sufficient condition for local uniqueness.

Global Uniqueness. We now show that explosive dynamics are incompatible with equilibrium. Online Appendix C.3 shows that a sufficient condition for explosive dynamics to be inconsistent with equilibrium is for real debt to grow at a rate greater than r_t . But this follows from Equation (C.40) and $\phi_r < 0$.

C.9 Local Dynamics with Interest Payment Reaction Rule

Steady-State Invariance. Suppose the government follows the fiscal rule:

$$s_t = s^* + \phi_s(r_t b_t - r^* b^*) \quad (\text{C.50})$$

where $s^* = r^* b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$db_t = [r_t b_t - s_t]dt \quad (\text{C.51})$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{s^* - \phi_s r^* b^*}{b - \phi_s b^*} = \frac{s^*}{b} \quad (\text{C.52})$$

which shows that the steady-states are unchanged. Hence, there is no scope for this fiscal rule to eliminate steady-state multiplicity.

Local Dynamics. The dynamics of government debt are given by

$$db_t = (1 - \phi_s) (\mathbf{r}(\Omega_t) b_t - s^*) dt \quad (\text{C.53})$$

It follows that the stability properties of the two-steady states in the baseline case with $\phi_s = 0$ are reversed when $\phi_s > 1$.

D Model With Foreign Demand for Debt

We assume that there exists a foreign sector that is populated by a representative household. The foreign representative household derives utility over real consumption streams and real debt holdings in terms of US goods.⁴¹ Preferences over foreign consumption and bonds are given by

$$u(c_t, d_t) = \frac{c_t^{1-\gamma}}{1-\gamma} + \tilde{\zeta} \frac{d_t^{1-\theta}}{1-\theta}$$

with $\gamma \geq 0$ and $\theta \geq 0$. The parameter $\tilde{\zeta} > 0$ parameterizes the payoff derived from real bond holdings. Households' rate of time preference is $\tilde{\rho}$. We assume the foreign sector grows at the same rate g as the domestic economy, thereby allowing for the existence of a balanced growth path. The household's growth-adjusted discount rate is therefore $\rho := \tilde{\rho} - (1 - \gamma)g$. In addition, we define $r_t := i_t - \pi_t - g$.

The household's budget constraint in real and stationary terms is therefore

$$dd_t = [r_t d_t + y^f - c_t] dt \quad (\text{D.54})$$

⁴¹Concretely, the foreign sector derives utility from nominal bonds in dollars divided by the US price level: $B_t/P_t^U S$. This is equivalent to holding real debt in terms of the foreign sector good ($P_t^F B_t/(P_t^F B_t)$) where P_t^F is the foreign price level. The implicit assumption here is that the final good is tradable, so that the exchange rate is constant.

where $y^f > 0$ is the foreign household's endowment of the consumption good. Foreign real consumption and real debt holdings must satisfy the following Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma} \left(r_t - \rho + \tilde{\zeta} \frac{d_t^{-\theta}}{c_t^{-\gamma}} \right)$$

In steady-state, the aggregate consumption of the foreign sector is y^f . Hence, for a given interest rate r^* , we must have

$$0 = r^* - \rho + \frac{\tilde{\zeta}}{y^f} d^{-\theta}$$

It follows that foreign sector demand for US government debt is given by

$$\mathbf{d}(r^*) = \left(\frac{\rho - r^*}{\zeta} \right)^{-\frac{1}{\theta}}$$

where we let $\zeta := \tilde{\zeta}/y^f$. Note that the relationship between foreign debt holdings and real interest rates can be written as:

$$\log d = \zeta + \frac{1}{\theta} \log(r^* - \rho)$$

so a large θ means more inelastic demand.

Consider the limits of this function: as $r^* \rightarrow -\infty$, $d \rightarrow 0$ and as $r^* \rightarrow \rho$, $d \rightarrow \infty$. We want to argue that when the foreign demand is inelastic enough (θ large), we can get rid of the high inflation steady-state. From Figure 5c, it is clear that what we need is the bond supply function $\mathbf{b}(r)$ to lie above the total bond demand function $\mathbf{a}(r) + \mathbf{d}(r)$ as $r \rightarrow -\infty$. This condition can be re-written as:

$$\lim_{r \rightarrow -\infty} \frac{\mathbf{b}(r)}{\mathbf{a}(r) + \mathbf{d}(r)} < 1$$

Or, equivalently

$$\lim_{b \rightarrow 0} \frac{r^g(b)}{r^h(b) + r^f(b)} < 1$$

where $r^g(b)$, $r^h(b)$, and $r^f(b)$ denotes the inverse debt demand functions for the government, domestic households, and the foreign sector, respectively. Substituting for

these functional forms yields

$$\lim_{b \rightarrow 0} \frac{\frac{s}{b}}{\mathbf{a}^{-1}(b) + (\rho - \zeta b^{-\theta})}$$

Online Appendix C.1 shows that there exists a finite real interest rate \underline{r} such that $\mathbf{a}(\underline{r}) = 0$. We may apply L'Hopital's rule to obtain

$$\lim_{b \rightarrow 0} \frac{-s}{b^2 (\kappa + \theta b^{-\theta-1})} < 1$$

where κ is a finite positive constant. This inequality is satisfied if and only if

$$-s < \theta b^{1-\theta}$$

Recall that $s < 0$ so the LHS is a positive finite number. As long as $\theta > 1$ the right-hand-side converges to infinity as $b \rightarrow 0$ which satisfies the inequality. Hence, the condition we need in order to obtain steady-state uniqueness is $\theta > 1$ which implies that the foreign demand has to be inelastic enough.

E Extended Model for Quantitative Analysis

E.1 Model With Borrowing

In this section, we describe how the model is consistent with a non-zero lower bound on real household assets and costly borrowing. Households face a borrowing limit expressed in real terms:

$$\frac{A_{jt}}{P_t} \geq \tilde{a}_t \tag{E.55}$$

In order for the borrowing constraint to be consistent with balanced growth, we assume that \tilde{a}_t grows at the rate of real output, $\tilde{a}_t = y_0 e^{gt} \underline{a}$ for some $\underline{a} < 0$. Note that this implies that $a_{jt} \geq \underline{a}$. Furthermore, we assume that borrowing is costly. Households face a wedge $\vartheta \geq 0$ on the real interest rate when borrowing, so that the interest rate they pay on debt is $\vartheta + r_t$. This borrowing wedge creates deadweight loss in output that is equal to the wedge multiplied by the amount of assets borrowed. This implies that aggregate consumption is slightly less than output (a difference around 0.3% of steady-state output in our calibration). The associated boundary

condition for the HJB equation (9) is:

$$\partial_a V_t(0, z) \geq (z - \tau_t(z) - (r_t + \vartheta)\underline{a})^{-\gamma} \quad (\text{E.56})$$

The government accumulation equation continues to follow equation (18), with the understanding that $b_t \geq 0$, so that the government can lend, but not borrow.

E.2 Model With Long-Term Debt

The government now issues two securities: short-term debt B_t^s that pays a nominal rate i_t , and long-term debt B_t^l . Long-term debt takes the form of depreciating consols that depreciate at a rate $\delta > 0$, and that yield a flow coupon payment of $\chi > 0$ as in [Cochrane \(2001\)](#). We let q_t denote the market value of this long-term bond. The government's budget constraint can be written as:

$$dB_t^s + q_t dB_t^l = [iB_t^s + (\chi - \delta q_t)B_t^l - P_t s_t] dt \quad (\text{E.57})$$

The intuition for this equation is as follows. The right-hand side is the government's nominal deficit that consists of the primary deficit $-P_t s_t$, interest payments on short-term debt iB_t^s , and coupon payments plus redemption of long-term debt $(\chi - \delta q_t)B_t^l$. Whenever the deficit is greater than zero, the government must issue additional debt. It can do so either by issuing additional short-term debt or by issuing additional long-term debt at the price of q_t .

Similarly, we may define the nominal short- and long-term debt holdings of household j at time t as A_{jt}^s and A_{jt}^l , respectively. The household budget constraint becomes

$$dA_{jt}^s + q_t dA_{jt}^l = [i_t A_{jt}^s + (\chi - \delta q_t) A_{jt}^l + (z_{jt} - \tau_t(z_{jt})) P_t y_t - P_t \tilde{c}_{jt}] dt \quad (\text{E.58})$$

We define the market value of total government debt outstanding as $B_t := B_t^s + q_t B_t^l$ and the total value of household assets as $A_{jt} := A_{jt}^s + q_t A_{jt}^l$. We also define de-trended real debt and assets as in the main text:

$$b_{jt} = \frac{B_t}{P_t} \quad \text{and} \quad a_{jt} = \frac{A_{jt}}{P_t} \quad (\text{E.59})$$

The next proposition demonstrates that there is maturity structure irrelevance for government debt: b_t and a_{jt} are the only state variables in this economy.

Proposition 4. *The household budget constraint follows (6) and the real government budget constraint follows (18) for $t > 0$. Moreover, the price of long-term debt satisfies the following differential equation for $t > 0$:*

$$\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t \tag{E.60}$$

Proof. See Not For Publication Appendix I □

This proof shows that an economy with long-term debt collapses into an economy with short-term debt in the absence of uncertainty. Equation (E.60) is an arbitrage relationship between short- and long-term debt. In equilibrium, households are indifferent between the two assets. Hence, long-term debt will only matter for inflation dynamics insofar there is an unanticipated change in nominal rates i_t . Equation (E.60) is a forward-looking equation.

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Not For Publication Appendix

F Derivation of Real Rate Functional

This section derives the real interest rate functional given in Equation (25). We start from the characterization of optimal consumption dynamics contained in the Online Appendix C.1. Namely, we use (C.9) and (C.13) to integrate across all households j :

$$\begin{aligned} \frac{d}{dt} \int_j c_t(a_j, z_j) dj &= \int_{j:u} \left(\partial_a c_t(a_j, z_j) s_t(a_j, z_j) + \partial_t c_t(a_j, z_j) + \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \right) dj \\ &\quad + \int_{j:c} \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(0, z') - c_t(0, z_j)] \end{aligned} \quad (\text{F.1})$$

where the $d\tilde{N}_j$ terms vanish by the exact law of large numbers (Duffie and Sun, 2007, 2012). The first integral on the right-hand side is over unconstrained households ($j : u$), while the second integral is over constrained households ($j : c$). Note that the above equation must be equal to zero, since $\int_j c_t(a_j, z_j) dj = 1$, by market clearing. Dividing by $u''(c_t(a_j, z_j))$ in (C.7) and using CRRA preferences, we obtain:

$$\begin{aligned} -\frac{1}{\gamma}(\rho - r)c_t(a_j, z_j) &= \sum_{z' \neq z_j} \lambda_{z_j z'} \frac{1}{u''(c_t(a_j, z_j))} [u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))] \\ &\quad + \partial_t c_t(a_j, z_j) + s_t(a_j, z_j) \partial_a c_t(a_j, z_j) \end{aligned} \quad (\text{F.2})$$

Integrating over all unconstrained agents and using (F.1) to substitute for $\partial_t c_t(a_j, z_j) + s_t(a_j, z_j) \partial_a c_t(a_j, z_j)$ yields

$$\begin{aligned} -\frac{1}{\gamma}(\rho - r) \int_{j:u} c_t(a_j, z_j) dj &= \int_{j:u} \sum_{z' \neq z_j} \lambda_{z_j z'} \frac{1}{u''(c_t(a_j, z_j))} [u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))] dj \\ &\quad - \int_j \sum_{z' \neq z_j} \lambda_{z_j z'} (c_t(a_j, z') - c_t(a_j, z_j)) dj \end{aligned} \quad (\text{F.3})$$

Moreover, constrained agents consume their current income z_j . Hence, adding and subtracting $-\frac{1}{\gamma}(\rho - r) \int_{j:c} z_j dj$ to the equation above and rearranging yields an expression for the interest rate:

$$r = \rho + \gamma \frac{\int_{j:u} \sum_{z' \neq z_j} \lambda_{z_j z'} \frac{[u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))]}{u''(c_t(a_j, z_j))} dj - \int_j \sum_{z' \neq z_j} \lambda_{z_j z'} (c_t(a_j, z') - c_t(a_j, z_j)) dj}{1 - \int_{j:c} z_j dj} \quad (\text{F.4})$$

Using CRRA utility, and the fact that $\lambda_{z_j z_j} - \sum_{z' \neq z_j} \lambda_{z_j z'}$, we may write the above expression as

$$r = \rho - \frac{\int_{j:u} c(a_j, z_j) \left[\sum_{z'} \lambda_{z_j z'} \left(\frac{c(a_j, z')}{c(a_j, z_j)} \right)^{-\gamma} \right] dj + \gamma \int_j c(a_j, z_j) \left[\sum_{z'} \lambda_{z_j z'} \frac{c(a_j, z')}{c(a_j, z_j)} \right] dj}{1 - \int_{j:c} z_j dj} \quad (\text{F.5})$$

Relative to the representative agent economy, the sum differs by two terms: the (i) marginal utility variation due to income risk for unconstrained agents, and (ii) consumption variation due to income risk for both constrained and unconstrained agents, multiplied by the coefficient of relative risk aversion. All of these terms are scaled by one minus the total income holdings of constrained agents (which is trivially less one since aggregate consumption is equal to one). The interest rate can be written as a functional in terms of aggregate states by replacing $c_t(a_{jt}, z_{jt})$ with $c(\omega_{jt} b_t, z_{jt}, \Omega_t)$. Equation (25) then follows directly.

G Household Problem with Diffusion Process

This section sets up an economy in which income follows a diffusion process. We derive as an auxiliary result that $r_t < \rho$ for all $t \geq 0$ in this economy.

Concretely, we assume that household income follows a diffusion process given by

$$dz_{jt} = \mu_z(z_{jt})dt + \sigma_z(z_{jt})dB_{jt} \quad (\text{G.6})$$

where B_{jt} is adapted Brownian motion, independent across j , and $\mu_z(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma_z(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^+$ are twice-differentiable functions. We further assume that (G.6) admits a stationary distribution. The household problem now satisfies the following

HJB equation:

$$\begin{aligned} \rho V_t(a, z) - \partial_t V_t(a, z) &= \max_c \frac{c^{1-\gamma}}{1-\gamma} + \partial_a V_t(a, z) [r_t a + z - \tau_t(z) - c] \\ &\quad + \mu_z \partial_z V_t(a, z) + \frac{1}{2} \sigma_z^2 \partial_{zz}^2 V_t(a, z), \end{aligned} \quad (\text{G.7})$$

together with the boundary condition $\partial_a V_t(0, z) \geq (z - \tau_t(z))^{-\gamma}$. A solution to the HJB equation alongside (12) solves the household problem. The associated KFE equation is:

$$\partial_t g_t(a, z) = -\partial_a [g_t(a, z) \varsigma_t(a, z)] - \partial_z [\mu_z(z) g_t(a, z)] + \frac{1}{2} \partial_{zz}^2 [\sigma_z^2(z) g_t(a, z)] \quad (\text{G.8})$$

Expected Consumption Dynamics. We now derive the expected consumption dynamics for unconstrained households. Following exactly the same steps outlined in Online Appendix C.1 for the case in which income follows a Poisson process, we can derive an Euler equation for unconstrained households:

$$\begin{aligned} (\rho - r_t) u'(c_t(a, z)) &= \mu_z(z) u''(c_t(a, z)) \partial_z c_t(a, z) \\ &\quad + \frac{1}{2} \sigma_z^2(z) (u''(c_t(a, z)) \partial_{zz}^2 c_t(a, z) + u'''(c_t(a, z)) (\partial_z c_t(a, z))^2) \\ &\quad + u''(c_t(a, z)) [\partial_t c_t(a, z) + \varsigma_t(a, z) \partial_a c_t(a, z)] \end{aligned} \quad (\text{G.9})$$

We can also use Ito's lemma on $c_t(a_{jt}, z_{jt})$ to obtain

$$\begin{aligned} dc_t(a_{jt}, z_{jt}) &= [\partial_t c_t(a_{jt}, z_{jt}) + \varsigma_t(a_{jt}, z_{jt}) \partial_a c_t(a_{jt}, z_{jt})] dt \\ &\quad + [\mu_z(z_{jt}) \partial_z c_t(a_{jt}, z_{jt}) + \frac{1}{2} \sigma_z^2(z_{jt}) \partial_{zz}^2 c_t(a_{jt}, z_{jt})] dt + \sigma_z(z_{jt}) \partial_z c_t(a_{jt}, z_{jt}) dB_{jt} \end{aligned} \quad (\text{G.10})$$

Taking expectations of the above equation, combining it with (G.9), and imposing that u is isoelastic with curvature parameter γ yields the expected consumption dynamics for unconstrained households:

$$\frac{\mathbb{E}_t[dc_{jt}]}{c_{jt} dt} = \frac{1}{\gamma} (r_t - \rho) + \frac{\gamma + 1}{2} \sigma_z^2(z_{jt}) \left(\frac{\partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2 \quad (\text{G.11})$$

Constrained households simply consume their income. Hence, their consumption dynamics are

$$dc_{jt} = [\mu_z(z_{jt})]dt + \sigma_z(z_{jt})dB_{jt} \quad (\text{G.12})$$

The expected consumption dynamics of constrained households are therefore given by

$$\frac{\mathbb{E}_t[dc_{jt}]}{dt} = \mu_z(z_{jt}) \quad (\text{G.13})$$

Derivation of Interest Rate Functional. Integrating over the consumption dynamics of unconstrained households and making use of the fact that

$$\int_j \frac{dc_{jt}}{dt} dj = 0$$

yields

$$\begin{aligned} 0 = & \int_{j:u} \frac{1}{\gamma} (r_t - \rho) c_{jt} dj + \int_{j:u} \frac{(\gamma + 1)}{2} c_t(a_{jt}, z_{jt}) \left(\frac{\sigma_z(z_{jt}) \partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2 dj \\ & + \int_{j:c} [\mu_z(z_{jt}) c_t(a_{jt}, z_{jt})] dj \end{aligned} \quad (\text{G.14})$$

where we have used (G.11) and (G.13). Finally, imposing market clearing $\int_j c_{jt} dt = 1$ yields

$$r_t = \rho - \frac{\frac{\gamma(\gamma+1)}{2} \int_{j:u} c_t(a_{jt}, z_{jt}) \left(\frac{\sigma_z(z_{jt}) \partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2 dj + \gamma \int_{j:c} [c_t(a_{jt}, z_{jt}) \mu_z(z_{jt})] dj}{1 - \int_{j:c} z_{jt} dj} \quad (\text{G.15})$$

Note that this implies that $r_t < \rho$ for all $t \geq 0$ (not just in steady-state) if no households are constrained, or if $\int_{j:c} \mu_z(z_{jt}) dj > 0$, so that constrained households expect their income to increase, on average. We may also write the formula analogously as the one in the main text for the Poisson income process (25):

$$0 = \frac{\mathcal{C}_t^u}{\gamma} (r_t - \rho) + \mathcal{C}_t^u \tilde{\mathbb{E}}_t^u \left[\frac{\gamma + 1}{2} \sigma_z^2(z) \left(\frac{\partial_z c_t(a, z)}{c_t(a, z)} \right)^2 - \mu_z(z) \right] + \tilde{\mathbb{E}}_t [\mu_z(z)] \quad (\text{G.16})$$

H Additional Details on Long-Run Anchoring

In this section, we demonstrate how the monetary authority can eliminate all dynamic equilibria that converge to the high inflation steady-state, leaving only a unique equilibrium that leads to the saddle-path stable, low-inflation steady-state. Concretely, suppose the monetary authority has the power to coordinate private sector beliefs about long-run inflation. Under such a setting we envisage two pillars of central bank policy: (i) a path or rule for short-term nominal interest rates i_t , and (ii) a long-run inflation target π^* . Whereas the interest rate is a policy tool that the central bank directly implements by intervening in appropriate markets or paying interest on reserves, the long-run inflation target is no more than an attempt to coordinate beliefs. If:

- (i) the long-run inflation target and the long-run nominal interest rate (π^*, i^*) are set to be consistent with the equilibrium real rate at the saddle-path steady state, $i^* - \pi^* - g = r_H^*$;
- (ii) fiscal policy follows a constant deficit policy or a passive interest payment reaction rule with $\phi_s < 1$, so that the high real rate, low inflation steady-state is saddle-path stable;
- (iii) private sector beliefs about long-run inflation are consistent with the central bank's target,

then there is a unique real equilibrium and the price-level and inflation are pinned down for all t . The third of these conditions is a big “if”, and there is no fundamental reason to expect it to hold. However the key point is that managing *long-run* inflation expectations is sufficient to pin down the price level and inflation in the *short-run*. If the central bank is successful at convincing the private sector to coordinate on a long-run inflation target, then this is sufficient to eliminate any indeterminacy about inflation at all points in time. Note that anchoring long-run inflation expectations at π^* does not assume away the issue of price-level determination in the short-run. Both the initial price level and subsequent inflation remain endogenous and depend on monetary policy, fiscal policy and private sector behavior.

Even with long-run inflation anchoring, fiscal policy remains an essential component of price-level determination. Coordinating long-run expectations only uniquely determines the price-level in the short-run if fiscal policy acts in a way that ensures

the saddle-path stability of the low-inflation steady state. Such fiscal policy settings are the same as those required for uniqueness in the case with persistent surpluses.

I Proof for the Model With Long-Term Debt

Proposition 5. *The household budget constraint follows (6) and the real government budget constraint follows (18) for $t > 0$. Moreover, the price of long-term debt satisfies the following differential equation for $t > 0$:*

$$\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t \quad (\text{I.17})$$

Proof. We define the auxiliary variable $u = A_{jt}^l$. Note that this implies $dA_{jt}^l = u$. Hence, the households HJB equation is given by:

$$\begin{aligned} & \tilde{\rho}V_t(A^l, A^s, z) - \partial_t V(A^l, A^s, z) = \\ & \max_{c,u} \frac{c^{1-\gamma}}{1-\gamma} + \tilde{s}_t \partial_{A^s} V_t(A^l, A^s, z) + \partial_{A^l} V_t(A^l, A^s, z)u + \sum_{z' \neq z} \lambda_{zz'} [V_t(A^l, A^s, z') - V_t(A^l, A^s, z)] \end{aligned}$$

where

$$\tilde{s}_t := i_t A^s + (\chi - \delta q_t) A^l + (z - \tau(z)) P_t y_t - P_t \tilde{c}_t - q_t u$$

The first-order condition with respect to u is given by:

$$q_t \partial_{A^s} V(A^l, A^s, z) = \partial_{A^l} V_t(A^l, A^s, z) \quad (\text{I.18})$$

We may differentiate with respect to time to obtain:

$$q_t \partial_{A^s, t}^2 V_t(A^l, A^s, z) + \partial_t q_t \partial_{A^s} V(A^l, A^s, z) = \partial_{A^l, t} V_t(A^l, A^s, z) \quad (\text{I.19})$$

The envelope condition for the HJB with respect to A^l is:

$$\tilde{\rho} \partial_{A^l} V_t - \partial_{t, A^l}^2 V_t = \tilde{s}_t \partial_{A^s, A^l}^2 V_t + (\chi - \delta q_t) \partial_{A^l} V_t + u \partial_{A^l}^2 V_t + \sum_{z' \neq z} \lambda_{zz'} [\partial_{A^l} V_t - \partial_{A^l} V_t] \quad (\text{I.20})$$

Similarly, the envelope condition for the HJB with respect to A^s is:

$$\tilde{\rho}\partial_{A^s}V_t - \partial_{t,A^s}^2V_t = \tilde{s}_t\partial_{A^s}^2V_t + i_t\partial_{A^s}V_t + u\partial_{A^l,A^s}^2V_t + \sum_{z' \neq z} \lambda_{zz'}[\partial_{A^s}V_t - \partial_{A^s}V_t] \quad (\text{I.21})$$

Multiplying (I.21) by q_t , subtracting Equation (I.20) from (I.21) and using (I.18) and (I.19) yields:

$$(q_t i_t - (\chi - \delta q_t) - \partial_t q_t)\partial_{A^l}V_t = 0 \quad (\text{I.22})$$

By market clearing, we must have $\partial_{A^l}V_t > 0$ (otherwise no long-term debt would be purchased in equilibrium). Hence, we have the arbitrage relationship:

$$\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t \quad (\text{I.23})$$

Differentiating $B_t = q_t B_t^l + i_t B_t^s$ and using the (E.57) yields (15), which can be written in real terms. This completes the proof. \square

J Supplement on Wealth Distribution and MPCs

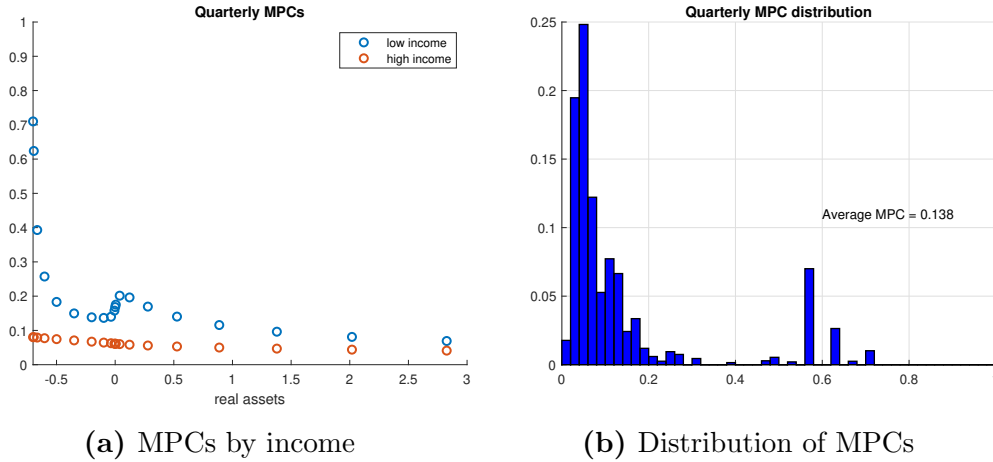
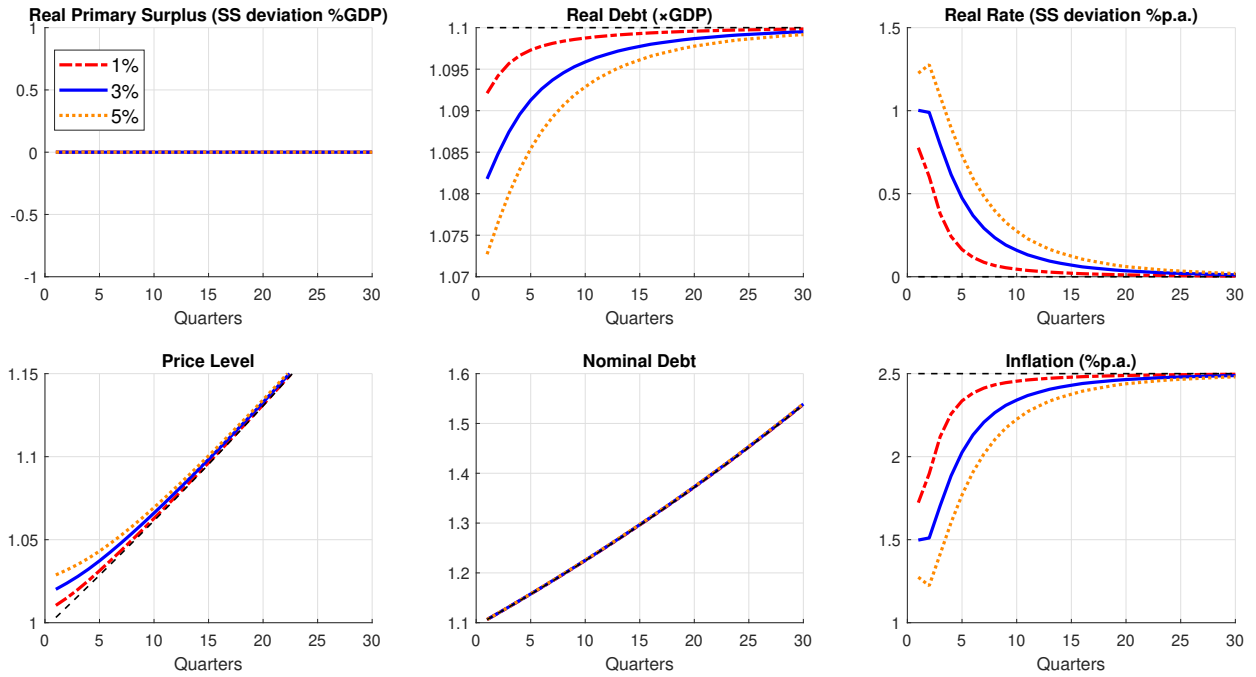


Figure 14: MPCs in the calibrated steady-state

This section provides some additional detail on the MPCs in the calibrated steady-state. Figure 14a shows the dependence of marginal propensities to consume on real assets, disaggregated by the highest and lowest income draws. The plotted MPCs are

Figure 15



Note: Impulse responses to a temporary increase in the wealth tax, with the proceedings distributed lump-sum, for various values of the wealth tax. In all experiments, the wealth tax is levied on the top 10% of the wealth distribution, the proceeds of which are redistributed lump-sum to the bottom 60%.

the quarterly marginal propensities to consume from an unanticipated \$500 income gain.

MPCs are not monotonically decreasing in real assets because there is a borrowing wedge. Households with zero assets therefore have a high marginal propensity to consume because of the discontinuous cost of borrowing (Kaplan and Violante, 2014). Note that the MPCs of high income households lie uniformly below the MPCs of low income households.

Figure 14b plots the distribution of MPCs in the calibrated steady-state. A large number of households have an MPC of around 0.15 and hold zero assets. The average MPC in the economy is 0.14, which is in line with commonly estimated values for marginal propensities to consume (Jappelli and Pistaferri, 2010).

K Inflationary Effects of Pure Redistribution

A comparison of the heterogeneous agent and representative agent economies in the preceding experiments suggests that redistribution itself has effects on the price level and inflation that are independent of the overall level of surpluses and nominal government debt. To emphasize the inflationary effects of redistribution, Figure 15 shows simulations from purely redistributive shocks. We consider one-time wealth taxes levied on the top 10% of the wealth distribution, the proceeds of which are redistributed lump-sum to the bottom 60%. Although these shocks do not entail any new issuance of government debt or any change in primary deficits, they do cause a period of inflation. The redistribution causes upward pressure on consumption because low-wealth households have higher average MPCs than high wealth households. Equilibrium is achieved through a period of higher real interest rates. The corresponding lower government revenues require a downward revaluation in real debt through a jump in the price level.

Inflationary Effects of Proportional Wealth Taxes. We contrast this experiment with another version of wealth taxation. Consider an economy where the government levies a proportional wealth tax at a rate of τ_b so that total primary surpluses are $s^* + \tau_b b_t$ (where s^* are surpluses net of revenue from the wealth tax). The real government budget constraint becomes:

$$db_t = [(r_t - \tau_b)b_t - s^*] dt. \tag{K.24}$$

The wealth tax appears in the household budget constraint in a similar fashion, as it increases the after-tax real rate paid to the government, $r_t - \tau_b$. Changes in τ_b therefore only affect the inflation rate through the Fisher equation, but otherwise leave the real economy and the initial price level unchanged.

L Endogenous Output

In this subsection, we outline an economy in which labor is a variable input in production. Next, we discuss how endogenous output affects price level and inflation dynamics in response to unanticipated shocks.

L.1 Set-Up

Households. The set-up of the household problem closely follows that of the main text. However, we assume that households choose real consumption flows \tilde{c}_{jt} and hours worked ℓ_{jt} to maximize

$$\mathbb{E}_0 \int e^{-\rho t} \left[\frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma} - \phi_t^{1-\gamma} \frac{\ell_{jt}^{1+\psi}}{1+\psi} \right] dt \quad (\text{L.25})$$

where the expectation is taken with respect to households' efficiency units of labour z_{jt} . The exponent $\psi > 0$ is the inverse of the Frisch elasticity of labor supply. The term ϕ_t is a time-varying constant that augments the labor disutility in order to allow the economy to be consistent with balanced growth when $\gamma \neq 1$. Concretely, we assume that

$$\phi_t = \tilde{\phi} e^{gt} \quad (\text{L.26})$$

where $\tilde{\phi} > 0$ and $g > 0$ is the growth rate of the economy. This formulation implies that a stationary equilibrium exists. Moreover, the distribution of hours across households is constant in the stationary equilibrium.⁴² The households nominal budget constraint therefore satisfies

$$dA_{jt} = [i_t A_{jt} + (1 - \tau_{1t}) z_{jt} P_t w_t \ell_{jt} - P_t \tilde{c}_{jt} + P_t \tau_{0t}] dt \quad (\text{L.27})$$

where w_t is the real wage rate for effective labor services at time t , τ_{0t} is a lump-sum payment and τ_{1t} is a constant proportional tax rate. We assume that τ_{0t} grows at a rate $g > 0$ in order to ensure that a stationary equilibrium exists:

$$\tau_{0t} = \tilde{\tau}_0 e^{gt} \quad (\text{L.28})$$

Finally, the stochastic process for z_{jt} and the definition of de-trended real variables for the evolution of real debt are identical to those of the main text.

⁴²We intentionally assume separability between hours and consumption in the instantaneous utility function so as to maximize comparability between the economy with endogenous output presented in this subsection and the endowment economy presented in the main text. In particular, the endowment economy can be closely approximated for large ψ and a given calibrated $\tilde{\phi}$. We note, however, that preferences by King et al. (1988) leave the key mechanisms unaffected.

Firms. We assume that perfectly competitive firms hire labor to produce output y_t with the constant returns to scale (CRS) production function

$$y_t = \Theta_t L_t \tag{L.29}$$

where Θ_t is aggregate total factor productivity that grows at a rate $g > 0$ and L_t are total effective hours:

$$L_t := \int_j z_{jt} \ell_{jt} dj \tag{L.30}$$

CRS implies that the real wage rate w_t is equal to Θ_t for all $t \geq 0$.

Government. The dynamics for government debt are given by

$$dB_t = [i_t B_t - s_t P_t y_t] dt \tag{L.31}$$

where s_t is the ratio of primary surpluses to output and is determined by the τ_{0t} and τ_{1t} as

$$s_t = \frac{\tau_{0t}}{y_t} + \int_{j \in [0,1]} \tau_{1t} w_t z_{jt} \ell_{jt} dj \tag{L.32}$$

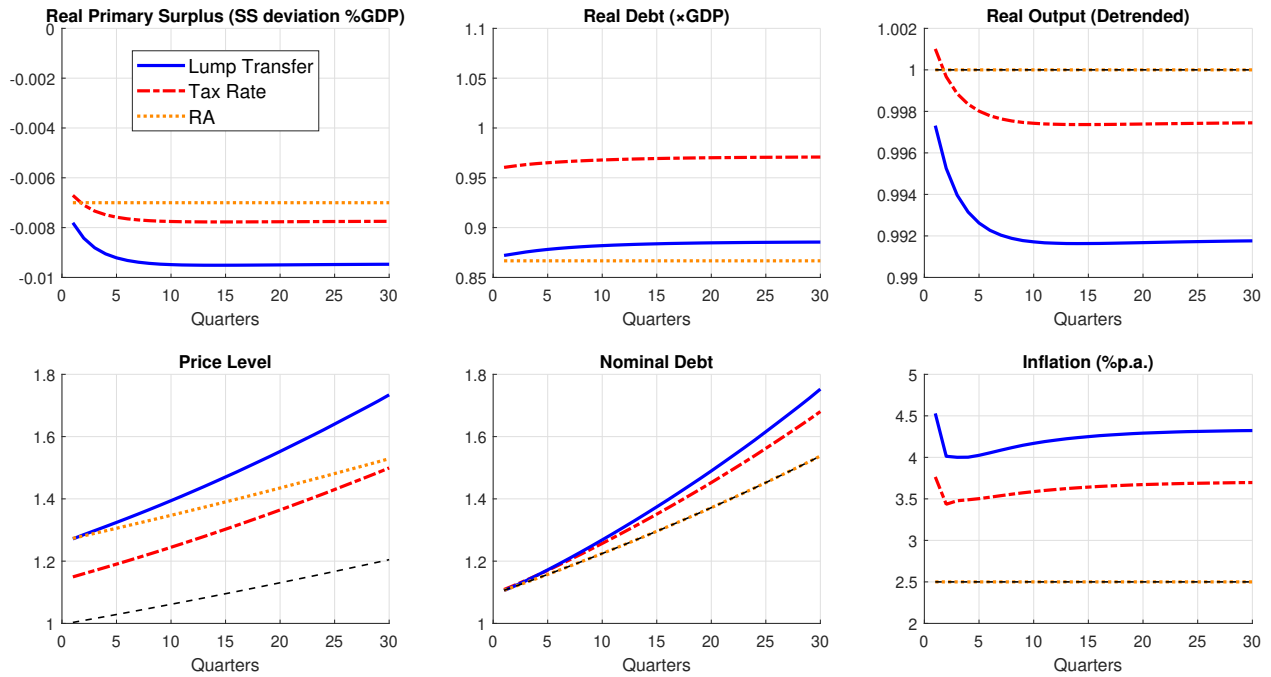
De-trended real government debt then follows

$$db_t = [r_t b_t - s_t] dt \tag{L.33}$$

We do not consider unanticipated changes in the nominal rate in this section. Consequently, we assume an interest rate peg $i_t = i^*$ without loss of generality in analyzing real dynamics.

Calibration. Our calibration sets $\psi = 2$, so that the intensive-margin Frisch elasticity of labor supply is equal to one-half, in line with the recommendation of [Chetty et al. \(2011\)](#). Moreover, we calibrate $\tilde{\phi}$ so as to set total hours worked equal to unity. Allowing labor to adjust on the intensive margin provides additional insurance to households. As such, the discount rate increases to 6.1% p.a. (relative to 2.8% p.a. from the calibration in the main text) in order to match a debt-to-annual GDP ratio of 1.10. The values for the remaining parameters remain unchanged from [Table 1](#).

Figure 16



Note: Impulse responses to a permanent expansion in primary deficits in the economy with endogenous output. The dotted orange line shows the effects of a permanent reduction in surpluses in the Representative Agent model due to a change in transfers. The solid blue line labelled “Lump Sum” illustrates the dynamics following an expansion of lump sum transfers. The dashed red line labelled “Tax Rate” illustrates the dynamics following a tax cut. In all experiments, deficits increase by 0.7% of pre-shock GDP.

L.2 Quantitative Exercise

We consider the economy’s response to an increase in deficits. First, we consider the economy’s response to a permanent change in $\tilde{\tau}_{0t}$ from 0.333 to 0.340, keeping τ_1 fixed. Second, we consider a permanent change in τ_{1t} from 0.300 to 0.307, keeping τ_{0t} fixed. These changes amount to a change in deficits from 3.3% to 4% of GDP, if output was unchanged (in line with the analysis of Section 5.5).

An increase in deficits due to a tax cut results in a smaller jump in the initial price level, relative to the transfer expansion case. The main reason is that lower taxation increases the labor supply (whereas a transfer expansion lowers it). The corresponding rise in output raises tax revenues and attenuates the long-run increase

in primary deficits relative to the transfer expansion.⁴³

In both economies, however, real output eventually *declines* relative to the representative agent benchmark. In order to understand this result, consider the tax cut experiment. There are two forces that contribute to an increase in labor supply. First, the tax cut directly raises the return to working, as explained above. Second, households in the new steady-state hold lower amounts of wealth, on average. This gives rise to positive wealth effects that also expands total hours worked. However, the new steady-state features a lower long-run real rate – a force only present in the heterogeneous agent economy. The reduction in the real rate increases consumption state-by-state due to the intertemporal savings motive, thereby reducing total hours worked. This last force is sufficiently strong that it counteracts the positive effect on output due to the lower tax rate and the change in the wealth distribution. Consequently, in the long-run output falls and deficits rise relative to the representative agent economy.

⁴³The tax cut also increases precautionary motives by amplifying the volatility of post-tax earnings, in line with the reasoning of Section 5.5. The real interest rate therefore decreases relatively less. Since the government now finances its debt at a higher cost, this a force that contributes to a larger initial jump in the price level. However, this mechanism is dominated by labor-supply channel.

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